1. Compute the definite integral:

\[ \int_1^5 \left( \frac{17}{x} + 3 \right) \, dx \]

**Solution:** Using the Fundamental Theorem of Calculus Part I, the value of the integral is:

\[
\int_1^5 \left( \frac{17}{x} + 3 \right) \, dx = \left[ 17 \ln |x| + 3x \right]_1^5 \\
= [17 \ln 5 + 3(5)] - [17 \ln |1| + 3(1)] \\
= 17 \ln 5 + 15 - 0 - 3 \\
= 17 \ln 5 + 12
\]
2. Consider the function \( f(x) = 2x - x^2 \) on the interval \([0, 2]\). Compute the trapezoid and midpoint approximations \( T_2 \) and \( M_2 \).

**Solution:** The length of each subinterval of \([0, 2]\) is:

\[
\Delta x = \frac{b - a}{N} = \frac{2 - 0}{2} = 1
\]

The trapezoid approximation \( T_2 \) is:

\[
T_2 = \frac{\Delta x}{2} \left[ f(0) + 2f(1) + f(2) \right]
= \frac{1}{2} \left[ (2 \cdot 0 - 0^2) + 2(2 \cdot 1 - 1^2) + (2 \cdot 2 - 2^2) \right]
= \frac{1}{2} [0 + 2 + 0]
= 1
\]

The midpoint approximation \( M_2 \) is:

\[
M_2 = \Delta x \left[ f \left( \frac{1}{2} \right) + f \left( \frac{3}{2} \right) \right]
= 1 \cdot \left[ \left( 2 \cdot \frac{1}{2} - \left( \frac{1}{2} \right)^2 \right) + \left( 2 \cdot \frac{3}{2} - \left( \frac{3}{2} \right)^2 \right) \right]
= \frac{3}{4} + \frac{3}{4}
= \frac{3}{2}
\]
3. The region enclosed by the graphs of the functions $y = x$ and $y = \sqrt{x}$ from $x = 0$ to $x = 1$ is rotated about the $y$-axis. Compute the volume of the resulting solid.

**Solution:** We will use the Shell Method to compute the volume. The formula is:

$$V = 2\pi \int_a^b x (\text{top} - \text{bottom}) \, dx$$

where the top curve is $y = \sqrt{x}$, the bottom curve is $y = x$, the interval is $0 \leq x \leq 1$. Therefore, the volume is:

$$V = 2\pi \int_0^1 x (\sqrt{x} - x) \, dx$$

$$= 2\pi \int_0^1 (x^{3/2} - x^2) \, dx$$

$$= 2\pi \left[ \frac{2}{5} x^{5/2} - \frac{1}{3} x^3 \right]_0^1$$

$$= 2\pi \left[ \frac{2}{5} - \frac{1}{3} \right]$$

$$= \frac{2\pi}{15}$$
Math 181, Exam 1, Study Guide 2
Problem 4 Solution

4. Compute the following integrals:
\[ \int \sin^2 x \cos^3 x \, dx \quad \int \frac{1}{\sqrt{4 - x^2}} \, dx \]

Solution: The first integral is computed by rewriting the integral using the Pythagorean Identity \( \cos^2 x + \sin^2 x = 1 \).

\[
\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx \\
= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx \\
= \int (\sin^2 x - \sin^4 x) \cos x \, dx
\]

Now let \( u = \sin x \). Then \( du = \cos x \, dx \) and we get:

\[
\int \sin^2 x \cos^3 x \, dx = \int (\sin^2 x - \sin^4 x) \cos x \, dx \\
= \int (u^2 - u^4) \, du \\
= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\
= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C
\]

The second integral is computed using the \( u \)-substitution method. Let \( u = \frac{1}{2}x \). Then \( du = \frac{1}{2} \, dx \) \( \Rightarrow \) \( 2 \, du = dx \) and \( x = 2u \). Substituting these into the integral and evaluating we get:

\[
\int \frac{1}{\sqrt{4 - x^2}} \, dx = \int \frac{1}{\sqrt{4 - (2u)^2}} (2 \, du) \\
= 2 \int \frac{1}{\sqrt{4 - 4u^2}} \, du \\
= 2 \int \frac{1}{\sqrt{4}\sqrt{1 - u^2}} \, du \\
= \frac{1}{\sqrt{1 - u^2}} \, du \\
= \arcsin u + C \\
= \arcsin \left( \frac{1}{2}x \right) + C
\]
Math 181, Exam 1, Study Guide 2
Problem 5 Solution

5. Compute the following integrals:

\[ \int \frac{x}{\sqrt{x - 2}} \, dx \quad \int \arctan x \, dx \]

**Solution:** The first integral is computed using the \( u \)-substitution method. Let \( u = x - 2 \). Then \( du = dx \) and \( x = u + 2 \). Substituting these into the integral and evaluating we get:

\[
\int \frac{x}{\sqrt{x - 2}} \, dx = \int \frac{u + 2}{\sqrt{u}} \, du = \int (u^{1/2} + 2u^{-1/2}) \, du = \frac{2}{3}u^{3/2} + 4u^{1/2} + C = \frac{2}{3}(x - 2)^{3/2} + 4(x - 2)^{1/2} + C
\]

The second integral is computed using Integration by Parts. Let \( u = \arctan x \) and \( v' = 1 \). Then \( u' = \frac{1}{x^2 + 1} \) and \( v = x \). Using the Integration by Parts formula:

\[
\int uv' \, dx = uv - \int u'v \, dx
\]

we get:

\[
\int \arctan x \, dx = x \arctan x - \int \frac{x}{x^2 + 1} \, dx
\]

The integral on the right hand side is computed using the \( u \)-substitution \( u = x^2 + 1 \). Then \( du = 2x \, dx \) \( \Rightarrow \) \( \frac{1}{2} \, du = x \, dx \) and we get:

\[
\int \arctan x \, dx = x \arctan x - \int \frac{x}{x^2 + 1} \, dx = x \arctan x - \int \frac{1}{x^2 + 1} \cdot x \, dx = x \arctan x - \frac{1}{2} \int \frac{1}{u} \, du = x \arctan x - \frac{1}{2} \ln |u| + C = x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C
\]
Math 181, Exam 1, Study Guide 2
Problem 6 Solution

6. Compute the following integrals:

\[ \int x^3 \sin(x^2) \, dx, \quad \int \frac{1}{x^2 + x - 6} \, dx \]

**Solution:** The first integral is computed using the \( u \)-substitution method. Let \( u = x^2 \).

Then \( du = 2x \, dx \Rightarrow \frac{1}{2} \, du = x \, dx \) and we get:

\[
\int x^3 \sin(x^2) \, dx = \int x^2 \sin(x^2) \, x \, dx \\
= \int u \sin u \left( \frac{1}{2} \, du \right) \\
= \frac{1}{2} \int u \sin u \, du
\]

We now use Integration by Parts to evaluate the above integral. Let \( w = u \) and \( v' = \sin u \).

Then \( w' = 1 \) and \( v = -\cos u \). Using the Integration by Parts formula:

\[
\int wv' \, du = vw - \int w'v \, du
\]

we get:

\[
\int u \sin u \, du = u(-\cos u) - \int 1 \cdot (-\cos u) \, du \\
= -u \cos u + \int \cos u \, du \\
= -u \cos u + \sin u + C
\]

Therefore,

\[
\int x^3 \sin(x^2) \, dx = \frac{1}{2} \int u \sin u \, du \\
= \frac{1}{2} (-u \cos u + \sin u) + C \\
= -\frac{1}{2} u \cos u + \frac{1}{2} \sin u + C
\]

\[
= -\frac{1}{2} x^2 \cos(x^2) + \frac{1}{2} \sin(x^2) + C
\]

1
The second integral is computed using Partial Fraction Decomposition. Factoring the denominator and decomposing we get:

$$\frac{1}{x^2 + x - 6} = \frac{1}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2}$$

Multiplying the equation by \((x + 3)(x - 2)\) we get:

$$1 = A(x - 2) + B(x + 3)$$

Next we plug in two different values of \(x\) to get a system of two equations in two unknowns \((A, B)\). Letting \(x = -3\) and \(x = 2\) we get:

\[
\begin{align*}
  x = -3 : & \quad 1 = A(-3 - 2) + B(-3 + 3) \quad \Rightarrow \quad A = -\frac{1}{5} \\
  x = 2 : & \quad 1 = A(2 - 2) + B(2 + 3) \quad \Rightarrow \quad B = \frac{1}{5}
\end{align*}
\]

Plugging these values of \(A\) and \(B\) back into the decomposed equation and integrating we get:

$$\int \frac{1}{x^2 + x - 6} \, dx = \int \left( -\frac{1}{5} \frac{1}{x + 3} + \frac{1}{5} \frac{1}{x - 2} \right) \, dx$$

$$= -\frac{1}{5} \ln |x + 3| + \frac{1}{5} \ln |x - 2| + C$$
Math 181, Exam 1, Study Guide 2
Problem 7 Solution

7. Compute the following integrals:

\[ \int \frac{1}{x^2+2x+3} \, dx, \quad \int x^6 \ln x \, dx \]

Solution: To compute the first integral, we will first complete the square in the denominator.

\[ \int \frac{1}{x^2+2x+3} \, dx = \int \frac{1}{(x+1)^2+2} \, dx \]

Now let \( u = x + 1 \). Then \( du = dx \) and we get:

\[ \int \frac{1}{(x+1)^2+2} \, dx = \int \frac{1}{u^2+2} \, du \]

Now let \( u = \sqrt{2}v \). Then \( du = \sqrt{2} \, dv \) and we get:

\[ \int \frac{1}{u^2+2} \, du = \int \frac{1}{(\sqrt{2}v)^2+2} \left( \sqrt{2} \, dv \right) \]

\[ = \sqrt{2} \int \frac{1}{2v^2+2} \, dv \]

\[ = \frac{\sqrt{2}}{2} \int \frac{1}{v^2+1} \, dv \]

\[ = \frac{\sqrt{2}}{2} \arctan v + C \]

\[ = \frac{\sqrt{2}}{2} \arctan \left( \frac{u}{\sqrt{2}} \right) + C \]

Therefore, the original integral is:

\[ \int \frac{1}{x^2+2x+3} \, dx = \frac{\sqrt{2}}{2} \arctan \left( \frac{x+1}{\sqrt{2}} \right) + C \]

The second integral is computed using Integration by Parts. Let \( u = \ln x \) and \( v' = x^6 \). Then \( u' = \frac{1}{x} \) and \( v = \frac{1}{7}x^7 \). Using the Integration by Parts formula:

\[ \int uv' \, dx = uv - \int u'v \, dx \]
we get:

\[
\int x^6 \ln x \, dx = (\ln x) \left( \frac{x^7}{7} \right) - \int \frac{1}{x} \cdot \frac{1}{7} x^7 \, dx
\]

\[
= \frac{1}{7} x^7 \ln x - \frac{1}{7} \int x^6 \, dx
\]

\[
= \frac{1}{7} x^7 \ln x - \frac{1}{49} x^7 + C
\]
8. Compute the following integrals:

\[ \int \cos (\sqrt{x}) \, dx, \quad \int x^2e^{2x} \, dx \]

**Solution**: To begin the solution of the first integral, we first use the \(u\)-substitution method. Let \( u = \sqrt{x} \). Then \( du = \frac{1}{2\sqrt{x}} \, dx \) \( \Rightarrow \) \( 2u \, du = dx \) and we get:

\[
\int \cos (\sqrt{x}) \, dx = \int \cos u \, (2u \, du) = 2 \int u \cos u \, du
\]

We now use Integration by Parts to evaluate the above integral. Let \( w = u \) and \( v' = \cos u \). Then \( w' = 1 \) and \( v = \sin u \). Using the Integration by Parts formula:

\[
\int wv' \, du = wv - \int w'v \, du
\]

we get:

\[
\int u \cos u \, du = u \sin u - \int \sin u \, du
\]

\[
\int u \cos u \, du = u \sin u + \cos u + C
\]

Therefore,

\[
\int \cos (\sqrt{x}) \, dx = 2 \int u \cos u \, du = 2 (u \sin u + \cos u) + C = 2u \sin u + \frac{1}{2} \cos u + C = 2\sqrt{x} \sin (\sqrt{x}) + 2 \cos (\sqrt{x}) + C
\]

The second integral is computed using Integration by Parts. Let \( u = x^2 \) and \( v' = e^{2x} \). Then \( u' = 2x \) and \( v = \frac{1}{2}e^{2x} \). Using the Integration by Parts formula:

\[
\int uv' \, dx = uv - \int u'v \, dx
\]
we get:

\[
\int x^2 e^{2x} \, dx = \frac{1}{2} x^2 e^{2x} - \int 2x \left( \frac{1}{2} e^{2x} \right) \, dx \\
= \frac{1}{2} x^2 e^{2x} - \int xe^{2x} \, dx
\]

A second Integration by Parts must be performed. Let \( u = x \) and \( v' = e^{2x} \). Then \( u' = 1 \) and \( v = \frac{1}{2} e^{2x} \). Using the Integration by Parts formula again we get:

\[
\int x^2 e^{2x} \, dx = \frac{1}{2} x^2 e^{2x} - \left[ \frac{1}{2} xe^{2x} - \frac{1}{2} \int e^{2x} \, dx \right] \\
= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} xe^{2x} + \frac{1}{4} e^{2x} + C
\]
9. Compute the area enclosed between the graphs $y = 1 - x^2$ and $y = 3 - 3x$.

Solution:

The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom}) \, dx$$

where the limits of integration are the $x$-coordinates of the points of intersection of the two curves. These are found by setting the y’s equal to each other and solving for $x$.

$$y = y$$
$$3 - 3x = 1 - x^2$$
$$x^2 - 3x + 2 = 0$$
$$(x - 1)(x - 2) = 0$$
$$x = 1, \ x = 2$$

From the graph we see that the top curve is $y = 1 - x^2$ and the bottom curve is $y = 3 - 3x$. 

\[1\]
Therefore, the area is:

\[
\text{Area} = \int_{1}^{2} \left[ (1 - x^2) - (3 - 3x) \right] \, dx \\
= \int_{1}^{2} (-2 + 3x - x^2) \, dx \\
= \left. \left[ -2x + \frac{3}{2} x^2 - \frac{1}{3} x^3 \right] \right|_{1}^{2} \\
= \left[ -2(2) + \frac{3}{2}(2)^2 - \frac{1}{3}(2)^3 \right] - \left[ -2(1) + \frac{3}{2}(1)^2 - \frac{1}{3}(1)^3 \right] \\
= \left[ -4 + 6 - \frac{8}{3} \right] - \left[ -2 + \frac{3}{2} - \frac{1}{3} \right] \\
= \frac{1}{6}
\]
10. A round hole of radius $b$ is drilled through the center of a hemisphere of radius $a$ ($a > b$). Find the volume of the portion of the sphere that remains.

Solution:

We use the Washer Method to compute the volume. The formula is:

$$V = \pi \int_{c}^{d} [(\text{right})^2 - (\text{left})^2] \, dy$$

where the right curve is $x = \sqrt{a^2 - y^2}$ and the left curve is $x = b$. The limits of integration are the $y$-coordinates of the points of intersection of the right and left curves. We find these by setting the $x$’s equal to each other and solving for $y$.

\begin{align*}
x &= x \\
b &= \sqrt{a^2 - y^2} \\
b^2 &= a^2 - y^2 \\
y^2 &= a^2 - b^2 \\
y &= \pm\sqrt{a^2 - b^2}
\end{align*}
The volume is then:

\[
V = \pi \int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} \left[ \left( \sqrt{a^2 - y^2} \right)^2 - b^2 \right] \, dy \\
= 2\pi \int_{0}^{\sqrt{a^2-b^2}} (a^2 - b^2 - y^2) \, dy \\
= 2\pi \left[ (a^2 - b^2)y - \frac{1}{3}y^3 \right]_{0}^{\sqrt{a^2-b^2}} \\
= 2\pi \left[ (a^2 - b^2)\sqrt{a^2-b^2} - \frac{1}{3} \left( \sqrt{a^2-b^2} \right)^3 \right] \\
= \frac{4\pi}{3} (a^2 - b^2)^{3/2}
\]