1. Compute the arc length of the graph of \( f(x) = \sqrt{9 - x^2} \) over \([0, 3]\).

**Solution:** The arc length can be easily found by recognizing that the graph of the function is a quarter circle of radius 3. Knowing that the arc length of a circle is \( 2\pi r \), the arc length of \( y = f(x) \) is

\[
\text{arc length} = \frac{1}{4} 2\pi(3) = \frac{3\pi}{2}.
\]

One can also resort to finding arc length via the formula

\[
L = \int_a^b \sqrt{1 + f'(x)^2} \, dx
\]

where

\[
f'(x) = -\frac{x}{\sqrt{9 - x^2}}
\]

The arc length is then

\[
L = \int_0^3 \sqrt{1 + \left(-\frac{x}{\sqrt{9 - x^2}}\right)^2} \, dx
\]

\[
L = \int_0^3 \sqrt{1 + \frac{x^2}{9 - x^2}} \, dx
\]

\[
L = \int_0^3 \sqrt{\frac{9 - x^2 + x^2}{9 - x^2}} \, dx
\]

\[
L = \int_0^3 \frac{3}{\sqrt{9 - x^2}} \, dx
\]

This integral may be solved using the trigonometric substitution \( x = 3 \sin \theta, \, dx = 3 \cos \theta \, d\theta \). Then \( \sqrt{9 - x^2} = 3 \cos \theta \) and we get

\[
L = \int_0^\pi/2 \frac{3}{3 \cos \theta} (3 \cos \theta \, d\theta)
\]

\[
L = \int_0^{\pi/2} 3 \, d\theta
\]

\[
L = \frac{3\pi}{2}
\]
2. Determine the limit of the sequence \( a_n = \frac{2n^2 + (0.3)^n}{3n^2 - n + 1} \).

**Solution:** We begin by multiplying the function by \( \frac{1}{n^2} \) divided by itself.

\[
\frac{2n^2 + (0.3)^n}{3n^2 - n + 1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{2 + (0.3)^n}{3 - \frac{1}{n} + \frac{1}{n^2}}
\]

Using the limit laws for quotients, sums, and differences we find that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2 + (0.3)^n}{3 - \frac{1}{n} + \frac{1}{n^2}}
\]

\[
= \lim_{n \to \infty} \left( \frac{2 + \lim_{n \to \infty} \frac{(0.3)^n}{n^2}}{\lim_{n \to \infty} 3 - \lim_{n \to \infty} \frac{1}{n} + \lim_{n \to \infty} \frac{1}{n^2}} \right)
\]

\[
= \frac{2 + 0}{3 - 0 + 0}
\]

\[
= \frac{2}{3}
\]

where we note that \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \) for \( p > 0 \) and \( \lim_{n \to \infty} r^n = 0 \) for \( 0 < r < 1 \).
Math 181, Exam 2, Fall 2011
Problem 3 Solution

3. Determine whether the improper integral converges, and if so, evaluate it:

(a) \( \int_1^\infty xe^{-x} \, dx \)

(b) \( \int_1^2 \frac{x}{x-1} \, dx \)

Solution:

(a) We evaluate the first integral by turning it into a limit calculation.

\[
\int_1^{+\infty} xe^{-x} \, dx = \lim_{R \to +\infty} \int_1^R xe^{-x} \, dx
\]

We use Integration by Parts to compute the integral. Let \( u = x \) and \( v' = e^{-x} \). Then \( u' = 1 \) and \( v = -e^{-x} \). Using the Integration by Parts formula we get:

\[
\int_a^b uv' \, dx = \left[ uv \right]_a^b - \int_a^b u'v \, dx
\]

\[
\int_1^R xe^{-x} \, dx = \left[ -xe^{-x} \right]_1^R - \int_1^R (-e^{-x}) \, dx
\]

\[
= \left[ -xe^{-x} \right]_1^R + \int_1^R e^{-x} \, dx
\]

\[
= \left[ -xe^{-x} \right]_1^R + \left[ -e^{-x} \right]_1^R
\]

\[
= \left[ -Re^{-R} + 1 \cdot e^{-1} \right] + \left[ -e^{-R} + e^{-1} \right]
\]

\[
= -\frac{R}{e^R} + \frac{1}{e} - \frac{1}{e^R} + \frac{1}{e}
\]

\[
= -\frac{R}{e^R} - \frac{1}{e^R} + \frac{2}{e}
\]
We now take the limit of the above function as $R \to +\infty$.

$$
\int_1^{+\infty} xe^{-x} \, dx = \lim_{R \to +\infty} \int_1^{R} xe^{-x} \, dx
= \lim_{R \to +\infty} \left( -\frac{R}{e^R} - \frac{1}{e^R} + \frac{2}{e} \right)
= -\lim_{R \to +\infty} \frac{R}{e^R} \lim_{R \to +\infty} \frac{1}{e^R} + \frac{2}{e}
= -\lim_{R \to +\infty} \frac{R}{e^R} - 0 + \frac{2}{e}
= \lim_{R \to +\infty} \frac{(R)'(e^R)'}{e^R} - 0 + \frac{2}{e}
= -\lim_{R \to +\infty} \frac{1}{e^R} - 0 + \frac{2}{e}
= -0 - 0 + \frac{2}{e}
= \frac{2}{e}
$$

(b) We begin by letting $u = x - 1$. Then $du = dx$ and the limits of integration become $u = 1 - 1 = 0$ and $u = 2 - 1 = 1$. Furthermore, since $u = x - 1$ we have $x = u + 1$. Making these substitutions we get

$$
\int_1^{2} \frac{x}{x-1} \, dx = \int_0^{1} \frac{u+1}{u} \, du = \int_0^{1} \left( 1 + \frac{1}{u} \right) \, du = \int_0^{1} 1 \, du + \int_0^{1} \frac{1}{u} \, du
$$

The first integral is proper and evaluates to 1. However, the second integral is improper and diverges because it is a $p$-integral of the form $\int_0^{1} \frac{1}{u^p} \, du$ where $p \geq 1$. Therefore, the given integral **diverges**.
4. State whether the given series is convergent or not. If convergent find its sum.

(a) \[ \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \]

(b) \[ \sum_{n=1}^{\infty} \frac{3^n}{2^n} \]

Solution:

(a) We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

\[ \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n \]

This is a convergent geometric series because \(|r| = \left| \frac{1}{4} \right| < 1\). We can now use the formula:

\[ \sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r} \]

where \( M = 1, \ c = 1, \) and \( r = \frac{1}{4} \). The sum of the series is then:

\[ \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n = \left( \frac{1}{4} \right)^1 \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{3} \]

(b) We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

\[ \sum_{n=1}^{\infty} \frac{3^n}{2^n} = \sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n \]

This is a divergent geometric series because \(|r| = \left| \frac{3}{2} \right| > 1\).
5. Find the values of $x$ for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{n}$$

Solution: We determine the radius of convergence using the Ratio Test.

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{3^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{3^n x^n} \right|$$

$$= \lim_{n \to \infty} \left| 3 \cdot \frac{n}{n+1} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n \to \infty} \left| 3 \cdot \left( \frac{n}{n+1} \right) x \right|$$

$$= \lim_{n \to \infty} \left| 3 \cdot \left( \frac{1}{1 + \frac{1}{n}} \right) \right|$$

$$= 3|x| \lim_{n \to \infty} \left( \frac{1}{1 + \frac{1}{n}} \right)$$

$$= 3|x|$$

In order to achieve convergence, it must be the case that $\rho = 3|x| < 1$. Therefore, $|x| < \frac{1}{3}$.

We must now check the endpoints. Plugging $x = \frac{1}{3}$ into the given power series we get:

$$\sum_{n=1}^{\infty} \frac{3^n \left( \frac{1}{3} \right)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a divergent $p$-series ($p = 1 \leq 1$). Plugging in $x = -\frac{1}{3}$ we get:

$$\sum_{n=1}^{\infty} \frac{3^n \left( -\frac{1}{3} \right)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

i.e. the alternating harmonic series, which converges by the Leibniz Test. Thus, the interval of convergence is:

$$-\frac{1}{3} \leq x < \frac{1}{3}$$