1. Compute the sums of the following series (do not show that they converge).

(a) \( \sum_{k=0}^{\infty} \frac{2^{k-1}}{32^k} \)

(b) \( \sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} \)

Solution:

(a) We begin by rewriting the series as follows:

\[
\sum_{k=0}^{\infty} \frac{2^{k-1}}{32^k} = \sum_{k=0}^{\infty} \frac{2^k \cdot \frac{1}{2}}{(32)^k} = \sum_{k=0}^{\infty} \frac{1}{9^k} = \sum_{k=0}^{\infty} \left( \frac{2}{9} \right)^k.
\]

Recognizing that this is a geometric series with \( a = \frac{1}{2} \) and \( r = \frac{2}{9} \) we know that the series converges because \(|r| < 1\) and that its sum is

\[
\sum_{k=0}^{\infty} \frac{1}{2} \left( \frac{2}{9} \right)^k = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{2}{9}} = \frac{9}{14}.
\]

(b) This is a telescoping series. The partial fraction decomposition of the \( n \)th term is

\[
\frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}.
\]

The \( N \)th partial sum of the series is

\[
S_N = \sum_{n=2}^{N} \left( \frac{1}{n-1} - \frac{1}{n+1} \right),
\]

\[
S_N = \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \cdots
\]

\[
+ \left( \frac{1}{N-4} - \frac{1}{N-2} \right) + \left( \frac{1}{N-3} - \frac{1}{N-1} \right) + \left( \frac{1}{N-2} - \frac{1}{N} \right) + \left( \frac{1}{N-1} - \frac{1}{N+1} \right),
\]

\[
S_N = 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}.
\]
The sum of the series is the limit of $S_N$ as $N \to \infty$. That is,

$$
\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \lim_{N \to \infty} S_N,
$$

$$
\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \lim_{N \to \infty} \left( 1 + \frac{1}{2} - \frac{1}{N} \right) - \frac{1}{N+1},
$$

$$
\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = 1 + \frac{1}{2} - 0 - 0
$$

$$
\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \frac{3}{2}
$$
2. For each sequence below, determine its limit or show that it diverges.

(a) \[ \left\{ \frac{5^{2n}}{2^{5n}} \right\} \]

(b) \[ \{(2n)^{1/n}\} \]

Solution:

(a) We begin by rewriting the nth term of the sequence as follows:

\[
\frac{5^{2n}}{2^{5n}} = \left(\frac{5^2}{2^5}\right)^n = \frac{25^n}{32^n} = \left(\frac{25}{32}\right)^n
\]

The sequence is geometric with \(|r| = \left|\frac{25}{32}\right| < 1\). Therefore, we know that it converges to 0. That is,

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(\frac{25}{32}\right)^n = 0
\]

(b) First, we notice that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (2n)^{1/n} \to \infty^0
\]

which is indeterminate. We resolve this indeterminacy by rewriting the function using the exponential of a logarithm. That is,

\[
(2n)^{1/n} = e^{\ln(2n)^{1/n}} = e^{\frac{1}{n} \ln(2n)}
\]

Therefore, the value of the limit is

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} e^{\frac{1}{n} \ln(2n)} = e^0 = 1.
\]

We used the fact that \(\ln(2n) \ll n\) as \(n \to \infty\) to evaluate the limit.
Problem 3 Solution

3. Compute the integral or show that it diverges.

(a) \( \int_{2}^{\infty} \frac{dx}{x^2 + 4} \)

(b) \( \int_{2}^{3} \frac{dx}{(x - 2)^{5/4}} \)

Solution:

(a) We begin by rewriting the integral as a limit.

\[
\int_{2}^{\infty} \frac{dx}{x^2 + 4} = \lim_{b \to \infty} \int_{2}^{b} \frac{dx}{x^2 + 4}
\]

Using the trigonometric substitution \( x = 2 \tan(\theta) \) one can show that an antiderivative of \( \frac{1}{x^2 + 4} \) is

\[
\int \frac{dx}{x^2 + 4} = \frac{1}{2} \arctan \left( \frac{x}{2} \right)
\]

Therefore, the value of the integral is

\[
\int_{2}^{\infty} \frac{dx}{x^2 + 4} = \lim_{b \to \infty} \int_{2}^{b} \frac{dx}{x^2 + 4} = \lim_{b \to \infty} \left[ \frac{1}{2} \arctan \left( \frac{b}{2} \right) - \frac{1}{2} \arctan \left( \frac{1}{2} \right) \right] = \frac{\pi}{8}.
\]

(b) We begin by letting \( u = x - 2 \) and \( du = dx \). The limits of integration then become \( u = 0 \) and \( u = 1 \) upon substituting the original limits into the equation \( u = x - 2 \). Therefore, the integral becomes

\[
\int_{2}^{3} \frac{dx}{(x - 2)^{5/4}} = \int_{0}^{1} \frac{du}{u^{5/4}}
\]

which we recognize as a \( p \)-integral with \( p = \frac{5}{4} \). Since \( p > 1 \) we know that the integral diverges.
4. Use the Trapezoid Rule with 3 subintervals to approximate \( \int_0^\pi \sin(x) \, dx \).

**Solution:** Since \( a = 0 \), \( b = \pi \), and \( N = 3 \) we know that

\[
\Delta x = \frac{b - a}{N} = \frac{\pi - 0}{3} = \frac{\pi}{3}.
\]

The Trapezoidal estimate is then

\[
T_3 = \frac{\Delta x}{2} \left[ f(0) + 2f \left( \frac{\pi}{3} \right) + 2f \left( \frac{2\pi}{3} \right) + f(\pi) \right],
\]

\[
T_3 = \frac{\pi}{3} \left[ \sin(0) + 2\sin \left( \frac{\pi}{3} \right) + 2\sin \left( \frac{2\pi}{3} \right) + \sin(\pi) \right],
\]

\[
T_3 = \frac{\pi}{6} \left[ 0 + 2 \left( \frac{\sqrt{3}}{2} \right) + 2 \left( \frac{\sqrt{3}}{2} \right) + 0 \right],
\]

\[
T_3 = \frac{\pi\sqrt{3}}{3}.
\]
5. Determine whether each of the following series converges or diverges. Indicate the method you are using.

(a) \( \sum_{k=1}^{\infty} \frac{k^2}{k^4 + 1} \)

(b) \( \sum_{k=1}^{\infty} \frac{k!}{k^k} \)

Solution:

(a) First, we note that

\[
0 \leq \frac{k^2}{k^4 + 1} \leq \frac{k^2}{k^4} = \frac{1}{k^2}
\]

for all \( k \). Furthermore, the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is a convergent \( p \)-series since \( p = 2 > 1 \). Thus, we know that \( \sum_{k=1}^{\infty} \frac{k^2}{k^4 + 1} \) converges by the Comparison Test.

(b) Due to the presence of the factorial, we know that the Ratio Test is the preferred convergence test. The value of \( r \), the limit of the ratio of consecutive terms as \( k \to \infty \), is

\[
r = \lim_{k \to \infty} \frac{a_{k+1}}{a_k},
\]

\[
r = \lim_{k \to \infty} \frac{(k + 1)!}{(k + 1)^{k+1}} \frac{k^k}{k!},
\]

\[
r = \lim_{k \to \infty} \frac{(k + 1)k!}{(k + 1)^k(k + 1)} \frac{k!}{k!},
\]

\[
r = \lim_{k \to \infty} \frac{k^k}{(k + 1)^k},
\]

\[
r = \lim_{k \to \infty} \left( \frac{k}{k + 1} \right)^k,
\]

\[
r = \lim_{k \to \infty} \frac{1}{\left( \frac{k+1}{k} \right)^k},
\]

\[
r = \lim_{k \to \infty} \frac{1}{\left( 1 + \frac{1}{k} \right)^k},
\]

\[
r = \lim_{k \to \infty} \frac{1}{e^k},
\]

\[r = \frac{1}{e}.\]
Therefore, since $r = \frac{1}{\epsilon} < 1$ we know that the series converges.