1. Compute the following sums

(a) \[ \sum_{n=3}^{\infty} \frac{2}{n(n-1)} \]

(b) \[ \sum_{n=1}^{\infty} \frac{2^{n+3}}{5 \cdot 7^{3n-2}} \]

Solution:

(a) This is a \textit{telescoping series}. To compute the sum, we decompose the summand as follows:

\[ \frac{2}{n(n-1)} = \frac{2}{n-1} - \frac{2}{n} \]

The \(N\)th partial sum of the series is

\[ S_N = \left( \frac{2}{2} - \frac{2}{3} \right) + \left( \frac{2}{3} - \frac{2}{4} \right) + \cdots + \left( \frac{2}{N-2} - \frac{2}{N-1} \right) + \left( \frac{2}{N-1} - \frac{2}{N} \right) \]

The sum collapses into the following:

\[ S_N = \frac{2}{2} - \frac{2}{N} \]

The sum of the series is the limit of \(S_N\) as \(N \to \infty\). That is,

\[ \sum_{n=3}^{\infty} \frac{2}{n(n-1)} = \lim_{N \to \infty} S_N \]

\[ \sum_{n=3}^{\infty} \frac{2}{n(n-1)} = \lim_{N \to \infty} \left( \frac{2}{2} - \frac{2}{N} \right) \]

\[ \sum_{n=3}^{\infty} \frac{2}{n(n-1)} = 1 \]

(b) This is a \textit{geometric series}. To begin, we rewrite the summand as follows:

\[ \frac{2^{n+3}}{5 \cdot 7^{3n-2}} = \frac{2^3}{5 \cdot 7^{3n-2}} \cdot \frac{2^n}{(7^3)^n} = \frac{8}{5 \cdot 49^{1-1}} \cdot \left( \frac{2}{7^3} \right)^n = \frac{392}{5} \cdot \left( \frac{2}{343} \right)^n \]

Using the fact that

\[ \sum_{n=N}^{\infty} ar^n = r^N \cdot \frac{a}{1-r}, \quad |r| < 1 \]
we have

\[
\sum_{n=1}^{+\infty} \frac{2^{n+3}}{5 \cdot 7^{3n-2}} = \left( \frac{2}{343} \right)^1 \cdot \frac{392}{5} \cdot \frac{341}{343}
\]

\[
\sum_{n=1}^{+\infty} \frac{2^{n+3}}{5 \cdot 7^{3n-2}} = \frac{784}{1705}
\]
Math 181, Exam 2, Spring 2013
Problem 2 Solution

2. Compute the integral \( \int_{-\infty}^{+\infty} \frac{dx}{x^2 + 4x + 5} \).

Solution: We begin by splitting the integral as follows:

\[
\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 4x + 5} = \int_{-\infty}^{-2} \frac{dx}{x^2 + 4x + 5} + \int_{-2}^{+\infty} \frac{dx}{x^2 + 4x + 5}
\]

We split the integral at \(-2\) because the denominator becomes \((x + 2)^2 + 1\) after completing the square. Letting \(u = x + 2\), \(du = dx\) then gives us the sum

\[
\int_{-\infty}^{0} \frac{du}{u^2 + 1} + \int_{0}^{+\infty} \frac{du}{u^2 + 1}
\]

Each integral has the same value due to the function \(f(u) = \frac{1}{u^2 + 1}\) being even, i.e. it has symmetry with respect to the \(y\)-axis. The second integral evaluates to

\[
\int_{0}^{+\infty} \frac{du}{u^2 + 1} = \lim_{b \to +\infty} \int_{0}^{b} \frac{du}{u^2 + 1} = \lim_{b \to +\infty} \left[ \tan^{-1}(b) - \tan^{-1}(0) \right] = \frac{\pi}{2}
\]

Thus, the value of the improper integral is

\[
\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 4x + 5} = \frac{\pi}{2} + \frac{\pi}{2} = \pi
\]
3. Determine whether the following series converge or not.

(a) \(\sum_{n=1}^{+\infty} \cos \left( \frac{1}{n} \right)\)

(b) \(\sum_{n=1}^{+\infty} \left( \frac{n}{5n+3} \right)^n\)

(c) \(\sum_{n=1}^{+\infty} \frac{\sin^2(n)}{n^2}\)

Solution:

(a) Since \(\lim_{n \to \infty} a_n = \lim_{n \to \infty} \cos \left( \frac{1}{n} \right) = \cos(0) = 1 \neq 0\), the series diverges by the **Divergence Test**.

(b) Since \(\rho = \lim_{n \to \infty} (a_n)^{1/n} = \lim_{n \to \infty} \frac{n}{5n+3} = \frac{1}{5} < 1\) the series converges by the **Root Test**.

(c) Using the fact that \(0 \leq \sin^2(n) \leq 1\) for all \(n\) we have

\[
0 \leq \frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}
\]

for all \(n \geq 1\) and that \(\sum \frac{1}{n^2}\) is a convergent \(p\)-series, we can say that the series \(\sum \frac{\sin^2(n)}{n^2}\) converges by the **Comparison Test**.
4. Determine whether the following integrals converge or not:

(a) \[ \int_0^4 \frac{1}{\sqrt{4-x}} \, dx \]

(b) \[ \int_2^{+\infty} \frac{1}{x(\ln x)^2} \, dx \]

Solution:

(a) Letting \( u = 4 - x \), \( du = -dx \) the integral transforms as follows:

\[
\int_0^4 \frac{1}{\sqrt{4-x}} \, dx = -\int_4^0 \frac{1}{\sqrt{u}} \, du \\
= \int_0^4 \frac{1}{\sqrt{u}} \, du \\
= \int_0^1 \frac{du}{\sqrt{u}} + \int_1^4 \frac{du}{\sqrt{u}}
\]

where the first integral on the right hand side above is known to be a convergent \( p \)-integral and has the value

\[
\int_0^1 \frac{du}{u^{1/2}} = \frac{1}{1 - \frac{1}{2}} = 2
\]

The second integral is proper and has the value

\[
\int_1^4 \frac{du}{\sqrt{u}} = 2\sqrt{u}\bigg|_1^4 = 2\sqrt{4} - 2\sqrt{1} = 2
\]

Thus, the improper integral converges and has the value

\[
\int_0^4 \frac{dx}{\sqrt{4-x}} = 2 + 2 = 4
\]

(b) Letting \( u = \ln(x) \) and \( du = \frac{1}{x} \, dx \) the integral transforms as follows:

\[
\int_2^{+\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln(2)}^{+\infty} \frac{du}{u^2} \\
= \int_{\ln(2)}^1 \frac{du}{u^2} + \int_1^{+\infty} \frac{du}{u^2}
\]
The second integral is a $p$-integral whose value is
\[
\int_1^{+\infty} \frac{du}{u^2} = \frac{1}{2 - 1} = 1
\]

The first integral is evaluated as follows
\[
\int_{\ln(2)}^1 \frac{du}{u^2} = \left[-\frac{1}{u}\right]_{\ln(2)}^1 = -1 + \frac{1}{\ln(2)}
\]

Thus, the improper integral converges and has the value
\[
\int_2^{+\infty} \frac{dx}{x(\ln x)^2} = 1 - 1 + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}
\]
5. Compute the limit of each sequence or show that the sequence diverges.

(a) \( \{a_n\} = \left\{ \sqrt[n]{n^2 + n + 3} \right\} \)

(b) \( \{b_n\} = \left\{ \frac{n + \sin n}{2n - \cos n} \right\} \)

Solution:

(a) To begin, we rewrite the function as

\[ \sqrt[n]{n^2 + n + 3} = (n^2 + n + 3)^{1/n} = \exp \left( \ln(n^2 + n + 3)^{1/n} \right) \]

where \( \exp(x) = e^x \), by definition. Using the logarithm rule \( \ln(x^n) = n \ln(x) \) we have

\[ \exp \left( \frac{1}{n} \ln(n^2 + n + 3) \right) = \exp \left( \frac{\ln(n^2 + n + 3)}{n} \right) \]

We now use Theorems and to find the value of the limit. That is,

\[ \lim_{n \to \infty} (n^2 + n + 3)^{1/n} = \exp \left( \lim_{x \to \infty} \frac{\ln(x^2 + x + 3)}{x} \right) \]

\[ = \exp \left( \lim_{x \to \infty} \frac{2x + 1}{x^2 + x + 3} \right) \]

\[ = \exp \left( \lim_{x \to \infty} \frac{2}{x} \right) \]

\[ = \exp(0) \]

\[ = 1 \]

(b) The Squeeze Theorem is appropriate here. We know that

\[ \frac{n - 1}{2n + 1} \leq \frac{n + \sin(n)}{2n - \cos(n)} \leq \frac{n + 1}{2n - 1} \]

for all \( n \geq 0 \) and that

\[ \lim_{n \to \infty} \frac{n - 1}{2n + 1} = \lim_{n \to \infty} \frac{n + 1}{2n - 1} = \frac{1}{2} \]

Thus, the sequence converges to \( \frac{1}{2} \).