1. Find the limit of the following sequences as \( n \to \infty \).

(a) \( a_n = \frac{3n^4 - n^3 + 2}{2n^4 + n^2 - 10} \)

(b) \( b_n = \frac{n + \sin(n)}{2n^2 - n + 1} \)

Solution:

(a) We proceed by multiplying the function by \( \frac{1}{n^4} \) divided by itself and then use the fact that \( \lim_{n \to \infty} \frac{c}{n^p} = 0 \) for any constant \( c \) and any positive number \( p \).

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3n^4 - n^3 + 2}{2n^4 + n^2 - 10} \cdot \frac{1}{n^4},
\]

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3 - \frac{1}{n} + \frac{2}{n^4}}{2 + \frac{1}{n^2} - \frac{10}{n^2}},
\]

\[
\lim_{n \to \infty} a_n = \frac{3 - 0 + 0}{2 + 0 - 0},
\]

\[
\lim_{n \to \infty} a_n = \frac{3}{2}.
\]

(b) We begin by multiplying the given function by \( \frac{1}{n^2} \) divided by itself.

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n + \sin(n)}{2n^2 - n + 1} \cdot \frac{1}{n^2},
\]

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\frac{1}{n} + \frac{\sin(n)}{n^2}}{2 - \frac{1}{n} + \frac{1}{n^2}}.
\]

We know that the limits of \( \frac{1}{n} \) and \( \frac{1}{n^2} \) as \( n \to \infty \) are both 0 using the fact that \( \lim_{n \to \infty} \frac{c}{n^p} = 0 \) for any constant \( c \) and any positive number \( p \).

We use the Squeeze Theorem to evaluate the limit of \( \frac{\sin(n)}{n^2} \) as \( n \to \infty \). To begin, we note that \(-1 \leq \sin(n) \leq 1\) for all \( n \). We then divide each part of the inequality by \( n^2 \) to obtain

\[
-\frac{1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}.
\]

The limits of \( -\frac{1}{n^2} \) and \( \frac{1}{n^2} \) as \( n \to \infty \) are both 0. Thus, the limit of \( \frac{\sin(n)}{n^2} \) as \( n \to \infty \) is also 0 by the Squeeze Theorem.
The value of the limit is then:

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\frac{1}{n} + \frac{\sin(n)}{n^2}}{2 - \frac{1}{n} + \frac{1}{n^2}}.
\]

\[
\lim_{n \to \infty} b_n = \frac{0 + 0}{2 - 0 + 0} = 0.
\]
2. Determine whether each series converges or diverges. Justify your answer.

(a) \[ \sum_{k=2}^{\infty} \frac{1}{k \ln k} \]

(b) \[ \sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k} \]

Solution:

(a) We will use the Integral Test to show that the integral diverges. The function \( f(x) = \frac{1}{x \ln x} \) is positive and decreasing for \( x \geq 2 \) and the value of the integral of \( f(x) \) on \([2, \infty)\) is:

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x \ln x} \, dx,
\]

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \left[ \ln(\ln x) \right]_{2}^{b},
\]

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \left[ \ln(\ln b) - \ln(\ln 2) \right],
\]

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \infty
\]

Thus, since the integral diverges we know that the series diverges by the Integral Test.

Note: The antiderivative of \( \frac{1}{x \ln x} \) was determined using the substitution \( u = \ln x, \) \( du = \frac{1}{x} \, dx. \)

(b) The series is alternating so we will use the Alternating Series Test to show that it converges. First, we note that \( f(k) = \frac{k^2}{2^k} \) is positive and decreasing for \( k \geq 1. \) Also,

\[
\lim_{k \to \infty} \frac{k^2}{2^k} = 0
\]

because we know that exponential functions grow much faster than polynomials. Therefore, the series converges by the Alternating Series Test.

Note: An alternative solution is to show that the series converges absolutely by testing the series \( \sum \frac{k^2}{2^k} \) using, for example, the Ratio Test.
3. Determine whether each improper integral converges or diverges. Justify your answer.

(a) \( \int_0^2 \frac{1}{\sqrt{x}(x-2)} \, dx \)

(b) \( \int_1^\infty \frac{\arctan x}{x^2} \, dx \)

Solution:

(a) The integrand is undefined at both limits of integration so we begin by splitting the integral into two integrals.

\[
\int_0^2 \frac{1}{\sqrt{x}(x-2)} \, dx = \int_0^1 \frac{1}{\sqrt{x}(x-2)} \, dx + \int_1^2 \frac{1}{\sqrt{x}(x-2)} \, dx
\]

An antiderivative for \( \frac{1}{\sqrt{x}(x-2)} \) is found by letting \( u = \sqrt{x}, \ u^2 = x, \) and \( 2 \, du = \frac{1}{\sqrt{x}} \, dx. \) Making these substitutions we get

\[
\int \frac{1}{\sqrt{x}(x-2)} \, dx = \int \frac{2}{u^2 - 2} \, du
\]

We now use the Method of Partial Fractions to evaluate the integral on the right hand side. Omitting the details of the decomposition, we end up with

\[
\frac{2}{u^2 - 2} = \frac{1}{\sqrt{2}} \frac{1}{u - \sqrt{2}} - \frac{1}{\sqrt{2}} \frac{1}{u + \sqrt{2}}.
\]

The antiderivative of \( \frac{1}{\sqrt{x(x-2)}} \) is then

\[
\int \frac{1}{\sqrt{x(x-2)}} \, dx = \int \frac{2}{u^2 - 2} \, du,
\]

\[
= \int \left( \frac{1}{u - \sqrt{2}} - \frac{1}{u + \sqrt{2}} \right) \, du,
\]

\[
= \frac{1}{\sqrt{2}} \int \left( \frac{1}{u - \sqrt{2}} - \frac{1}{u + \sqrt{2}} \right) \, du,
\]

\[
= \frac{1}{\sqrt{2}} \left( \ln |u - \sqrt{2}| - \ln |u + \sqrt{2}| \right),
\]

\[
= \frac{1}{\sqrt{2}} \left( \ln |\sqrt{x} - \sqrt{2}| - \ln |\sqrt{x} + \sqrt{2}| \right),
\]

\[
= \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{x} - \sqrt{2}}{\sqrt{x} + \sqrt{2}} \right|.
\]
Returning to the integral \( \int_{1}^{2} \frac{1}{\sqrt{x}(x-2)} \, dx \) we find that

\[
\int_{1}^{2} \frac{1}{\sqrt{x}(x-2)} \, dx = \lim_{b \to 2^-} \int_{1}^{b} \frac{1}{\sqrt{x}(x-2)} \, dx, \\
\int_{1}^{2} \frac{1}{\sqrt{x}(x-2)} \, dx = \lim_{b \to 2^-} \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{b} - \sqrt{2}}{\sqrt{b} + \sqrt{2}} \right| - \frac{1}{\sqrt{2}} \ln \left| \frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right|.
\]

The limit of the first term is \(-\infty\) because the term inside the natural logarithm tends to 0 as \( b \to 2^- \). The second term is constant so it will remain constant in the limit as \( b \to 2^- \). Therefore, the value of the integral is

\[
\int_{1}^{2} \frac{1}{\sqrt{x}(x-2)} \, dx = -\infty
\]

Since this integral diverges, the integral \( \int_{0}^{2} \frac{1}{\sqrt{x}(x-2)} \, dx \) diverges as well.

(b) We begin by rewriting the integral as a limit.

\[
\int_{1}^{\infty} \frac{\arctan x}{x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\arctan x}{x^2} \, dx
\]

We now focus on finding an antiderivative of \( \frac{\arctan x}{x^2} \) using Integration by Parts. Letting \( u = \arctan x \) and \( dv = \frac{1}{x^2} \, dx \) we get \( du = \frac{1}{1+x^2} \, dx \) and \( v = -\frac{1}{x} \). Using the Integration by Parts formula we find that

\[
\int u \, dv = uv - \int v \, du, \\
\int \frac{\arctan x}{x^2} \, dx = -\frac{\arctan x}{x} + \int \frac{1}{x(1+x^2)} \, dx.
\]

The integral on the right hand side is evaluated using the Method of Partial Fractions.

\[
\int \frac{1}{x(1+x^2)} \, dx = \int \left( \frac{1}{x} - \frac{x}{1+x^2} \right) \, dx,
\]

\[
\int \frac{1}{x(1+x^2)} \, dx = \ln |x| - \frac{1}{2} \ln |x^2 + 1|,
\]

\[
\int \frac{1}{x(1+x^2)} \, dx = \ln \left| \frac{x}{\sqrt{x^2 + 1}} \right|.
\]

Thus, an antiderivative of \( \frac{\arctan x}{x^2} \) is

\[
\int \frac{\arctan x}{x^2} \, dx = -\frac{\arctan x}{x} + \ln \left| \frac{x}{\sqrt{x^2 + 1}} \right|.
\]
We can now evaluate the improper integral.

\[
\int_{1}^{\infty} \frac{\arctan x}{x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\arctan x}{x^2} \, dx,
\]

\[
\int_{1}^{\infty} \frac{\arctan x}{x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{\arctan x}{x} + \ln \left| \frac{x}{\sqrt{x^2 + 1}} \right| \right]_{1}^{b},
\]

\[
\int_{1}^{\infty} \frac{\arctan x}{x^2} \, dx = \lim_{b \to \infty} \left[ \left( -\frac{\arctan b}{b} + \ln \left| \frac{b}{\sqrt{b^2 + 1}} \right| \right) - \left( -\frac{\arctan 1}{1} + \ln \left| \frac{1}{\sqrt{1^2 + 1}} \right| \right) \right],
\]

\[
\int_{1}^{\infty} \frac{\arctan x}{x^2} \, dx = \lim_{b \to \infty} \left( -\frac{\arctan b}{b} \right) + \lim_{b \to \infty} \sqrt{b^2 + 1} \rightarrow 1 + \frac{\pi}{4} - \ln \left( \frac{1}{\sqrt{2}} \right),
\]

\[
\int_{1}^{\infty} \frac{\arctan x}{x^2} \, dx = 0 + \ln |1| + \frac{\pi}{4} - \ln \left( \frac{1}{\sqrt{2}} \right),
\]

\[
\int_{1}^{\infty} \frac{\arctan x}{x^2} \, dx = \frac{\pi}{4} - \ln \left( \frac{1}{\sqrt{2}} \right).
\]

Since the integral evaluates to a number we say that it converges.
4. Evaluate the following integrals:

(a) \[ \int (\cos x)^{-1} \sin^3 x \, dx \]

(b) \[ \int \frac{1}{x^2 - 4x - 12} \, dx \]

Solution:

(a) We begin by rewriting \( \sin^3 x \) as \( \sin x \sin^2 x = \sin x (1 - \cos^2 x) \). Now let \( u = \cos x \) and \(-du = \sin x \, dx\). The integral is then transformed and evaluated as follows:

\[
\int (\cos x)^{-1} \sin^3 x \, dx = \int \frac{1}{\cos x} \cdot \sin x (1 - \cos^2 x) \, dx,
\]

\[
\int (\cos x)^{-1} \sin^3 x \, dx = - \int \frac{1}{u} \cdot (1 - u^2) \, du,
\]

\[
\int (\cos x)^{-1} \sin^3 x \, dx = - \int \left( \frac{1}{u} - u \right) \, du,
\]

\[
\int (\cos x)^{-1} \sin^3 x \, dx = - \ln |u| + \frac{1}{2} u^2 + C,
\]

\[
\int (\cos x)^{-1} \sin^3 x \, dx = - \ln |\cos x| + \frac{1}{2} \cos^2 x + C.
\]

(b) Using the Method of Partial Fractions we find that

\[
\int \frac{1}{x^2 - 4x - 12} \, dx = \int \left( \frac{\frac{1}{8}}{x - 6} - \frac{\frac{1}{8}}{x + 2} \right) \, dx,
\]

\[
\int \frac{1}{x^2 - 4x - 12} \, dx = \frac{1}{8} \ln |x - 6| - \frac{1}{8} \ln |x + 2| + C.
\]
5. Find the volume of the solid obtained by rotating about the $x$-axis the region enclosed by the graphs of $y = 2x - x^2$ and $y = x$.

**Solution:** We find the volume using the Washer Method.

$$V = \int_a^b \pi \left( \text{top}^2 - \text{bottom}^2 \right) \, dx$$

From the graph below we see that the bottom curve is $y = x$ and the top curve is $y = 2x - x^2$. The intersection points are determined by setting the two equations equal to one another and solving for $x$.

$$y = y,$$
$$x = 2x - x^2,$$
$$x^2 - x = 0,$$
$$x(x - 1) = 0,$$
$$x = 0, \ x = 1.$$

The volume is then

$$V = \int_0^1 \pi \left( (2x - x^2)^2 - x^2 \right) \, dx,$$
$$V = \pi \int_0^1 \left( 4x^2 - 4x^3 + x^4 - x^2 \right) \, dx,$$
$$V = \pi \int_0^1 \left( x^4 - 4x^3 + 3x^2 \right) \, dx,$$
$$V = \pi \left[ \frac{1}{5}x^5 - x^4 + x^3 \right]_0^1,$$
$$V = \pi \left( \frac{1}{5} - 1 + 1 \right),$$
$$V = \frac{\pi}{5}.$$
6. Find the power series representation centered at 0 for the following functions. Give the interval of convergence of the series.

(a)  \( f(x) = \frac{1}{(1-x)^2} \)

(b)  \( g(x) = x^2 e^{-x} \)

Solution:

(a) First we recognize that

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x}.
\]

Then, using the fact that the power series for \( \frac{1}{1-x} \) centered at 0 is

\[
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots,
\]

we obtain

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \left( 1 + x + x^2 + x^3 + \cdots \right),
\]

\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots,
\]

\[
\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}.
\]

The interval of convergence of the power series for \( \frac{1}{1-x} \) is \(-1 < x < 1\) so we know that the power series for \( \frac{1}{(1-x)^2} \) converges for the same values of \( x \). Upon checking the endpoints \( x = -1 \) and \( x = 1 \) we get the two series

\[
\sum_{k=1}^{\infty} k(-1)^{k-1} \quad \text{and} \quad \sum_{k=1}^{\infty} k
\]

which both diverge by the Divergence Test. Thus, the interval of convergence is \(-1 < x < 1\).
(b) Using the fact that the power series centered at 0 for $e^x$ is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

we obtain

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots.$$

Therefore, the power series for $g(x) = x^2 e^{-x}$ is

$$x^2 e^{-x} = x^2 \left( 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right),$$

$$x^2 e^{-x} = x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \cdots,$$

$$x^2 e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+2}}{k!}.$$

The interval of convergence of the power series for $e^x$ is $-\infty < x < \infty$. Thus, the interval of convergence of the power series for $e^{-x}$ is also $-\infty < x < \infty$. Multiplication by $x^n$ where $n$ is a positive integer does not change the interval of convergence. Thus, the interval of convergence for $x^2 e^{-x}$ is $-\infty < x < \infty$. 
7. Let \( f(x) = \cos(2x) - 1 + 2x^2 \).

(a) Find the first two non-zero terms in the Maclaurin series expansion of \( f \).

(b) Using the expansion found in step (a) compute the limit:

\[
\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4}
\]

Solution:

(a) Using the fact that the Maclaurin series for \( \cos(x) \) is

\[
\cos(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

we have

\[
\cos(2x) = \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{(2k)!},
\]

\[
\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots,
\]

\[
\cos(2x) = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots.
\]

Therefore, the first two non-zero terms in the Maclaurin series expansion of \( f \) are

\[
f(x) = \cos(2x) - 1 + 2x^2,
\]

\[
f(x) = \left( 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots \right) - 1 + 2x^2,
\]

\[
f(x) = \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots.
\]

(b) Using the expansion from part (a), we evaluate the limit as follows:

\[
\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4} = \lim_{x \to 0} \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots,
\]

\[
\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4} = \lim_{x \to 0} \left( \frac{2}{3} - \frac{4}{45}x^2 + \cdots \right),
\]

\[
\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4} = \frac{2}{3}.
\]
8. An equation of a curve in polar coordinates is given by 
\[ r = 2 \cos \theta, \quad 0 \leq \theta \leq 2\pi. \]

(a) Rewrite the equation in Cartesian coordinates. Sketch and identify the curve.

(b) Find the arc length of the curve using the integral formula.

Solution:

(a) To rewrite the equation in Cartesian coordinates we begin by multiplying both sides of the equation by \( r \) to get 
\[ r^2 = 2r \cos \theta. \]
Then, recognizing the fact that \( x^2 + y^2 = r^2 \) and that \( x = r \cos \theta \) we get 
\[ x^2 + y^2 = 2x \]
To take things a step further, we put the \( 2x \) to the left hand side and complete the square to get 
\[
\begin{align*}
  x^2 + y^2 &= 2x, \\
  x^2 - 2x + y^2 &= 0, \\
  (x - 1)^2 - 1 + y^2 &= 0, \\
  (x - 1)^2 + y^2 &= 1.
\end{align*}
\]
which we recognize is a circle centered at \((1,0)\) with radius 1. A plot of the curve is sketched below.

![Plot of a circle centered at (1,0) with radius 1.](image)
(b) The arc length formula for a curve $r = f(\theta)$ defined on the interval $\alpha \leq \theta \leq \beta$ in polar coordinates is

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta.$$  

In this case, the function is $f(\theta) = 2\cos \theta$ so that $f'(\theta) = -2\sin \theta$. The arc length is then

\[
\begin{align*}
L &= \int_{0}^{2\pi} \sqrt{(2\cos \theta)^2 + (-2\sin \theta)^2} \, d\theta, \\
L &= \int_{0}^{2\pi} \sqrt{4\cos^2 \theta + 4\sin^2 \theta} \, d\theta, \\
L &= \int_{0}^{2\pi} \sqrt{4(\cos^2 \theta + \sin^2 \theta)} \, d\theta, \\
L &= \int_{0}^{2\pi} \sqrt{4(1)} \, d\theta, \\
L &= \int_{0}^{2\pi} 2 \, d\theta, \\
L &= 2\theta \bigg|_{0}^{2\pi}, \\
L &= 4\pi.
\end{align*}
\]