Math 181, Final Exam, Spring 2010  
Problem 1 Solution

1. Integrate:  
(a) \( \int x \cos(3x) \, dx \)  
(b) \( \int \frac{\sin x}{\sqrt{3 - \cos x}} \, dx \)

Solution:

(a) We will evaluate the integral using Integration by Parts. Let \( u = x \) and \( v' = \cos(3x) \). Then \( u' = 1 \) and \( v = \frac{1}{3} \sin(3x) \). Using the Integration by Parts formula:

\[
\int uv' \, dx = uv - \int u'v \, dx
\]

we get:

\[
\int x \cos(3x) \, dx = \frac{1}{3} x \sin(3x) - \int \frac{1}{3} \sin(3x) \, dx
\]

\[
= \frac{1}{3} x \sin(3x) - \left[ -\frac{1}{9} \cos(3x) \right] + C
\]

\[
= \frac{1}{3} x \sin(3x) + \frac{1}{9} \cos(3x) + C
\]

(b) We will evaluate the integral using the \( u \)-substitution method. Let \( u = 3 - \cos x \). Then \( du = \sin x \, dx \) and we get:

\[
\int \frac{\sin x}{\sqrt{3 - \cos x}} \, dx = \int \frac{1}{\sqrt{u}} \, du
\]

\[
= \int u^{-1/2} \, du
\]

\[
= 2u^{1/2} + C
\]

\[
= 2\sqrt{3 - \cos x} + C
\]
2. Evaluate: 

\( (a) \int_0^1 \frac{dx}{4 - x^2} \quad (b) \int_1^e x^2 \ln x \, dx \)

Solution:

(a) We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

\[
\frac{1}{4 - x^2} = \frac{1}{(2 - x)(2 + x)} = \frac{A}{2 - x} + \frac{B}{2 + x}
\]

Next, we multiply the above equation by \((2 - x)(2 + x)\) to get:

\[
1 = A(2 + x) + B(2 - x)
\]

Then we plug in two different values for \(x\) to create a system of two equations in two unknowns \((A, B)\). We select \(x = 2\) and \(x = -2\) for simplicity.

\[
x = 2 : \quad A(2 + 2) + B(2 - 2) = 1 \quad \Rightarrow \quad A = \frac{1}{4}
\]

\[
x = -2 : \quad A(2 - 2) + B(2 + 2) = 1 \quad \Rightarrow \quad B = \frac{1}{4}
\]

Finally, we plug these values for \(A\) and \(B\) back into the decomposition and integrate.

\[
\int_0^1 \frac{1}{4 - x^2} \, dx = \int_0^1 \left( \frac{A}{2 - x} + \frac{B}{2 + x} \right) \, dx
\]

\[
= \int_0^1 \left( \frac{1}{2 - x} + \frac{1}{2 + x} \right) \, dx
\]

\[
= \left[ -\frac{1}{4} \ln |2 - x| + \frac{1}{4} \ln |2 + x| \right]_0^1
\]

\[
= \left[ -\frac{1}{4} \ln 1 + \frac{1}{4} \ln 3 \right] - \left[ -\frac{1}{4} \ln 2 + \frac{1}{4} \ln 4 \right]
\]

\[
= \frac{1}{4} \ln 3
\]

(b) We evaluate the integral using Integration by Parts. Let \(u = \ln x\) and \(v' = x^2\). Then \(u' = \frac{1}{x}\) and \(v = \frac{1}{3} x^3\). Using the Integration by Parts formula:

\[
\int_a^b u v' \, dx = [u v]_a^b - \int_a^b u' v \, dx
\]
we get:

\[
\int_1^e x^2 \ln x \, dx = \left[ (\ln x) \left( \frac{1}{3} x^3 \right) \right]_1^e - \int_1^e \frac{1}{x} \cdot \frac{1}{3} x^3 \, dx \\
= \left[ \frac{1}{3} x^3 \ln x \right]_1^e - \frac{1}{3} \int_1^e x^2 \, dx \\
= \left[ \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 \right]_1^e \\
= \left[ \frac{1}{3} e^3 \ln e - \frac{1}{9} e^3 \right] - \left[ \frac{1}{3} (1)^3 \ln 1 - \frac{1}{9} (1)^3 \right] \\
= \frac{2}{9} e^3 + \frac{1}{9}
\]
3. Determine whether each series converges or diverges. If the series converges, determine whether the convergence is absolute or conditional.

(a) \( \sum_{n=1}^{\infty} \frac{n^3}{2^n} \)

(b) \( \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n + 1}}{n^2} \)

(c) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}} \)

Solution:

(a) We use the Ratio Test to determine whether or not the series converges.

\[
\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^3 \cdot 2^n}{2^{n+1} \cdot n^3} = \lim_{n \to \infty} \frac{1}{2} \left( \frac{n+1}{n} \right)^3 = \frac{1}{2}
\]

Since \( \rho = \frac{1}{2} < 1 \), the series \( \sum_{n=1}^{\infty} \frac{n^3}{2^n} \) converges by the Ratio Test.

(b) The series is alternating so we check for absolute convergence by considering the series of absolute values:

\[
\sum_{n=1}^{\infty} \frac{|(-1)^n \sqrt{n + 1}|}{n^2} = \sum_{n=1}^{\infty} \frac{\sqrt{n + 1}}{n^2}
\]

We note that:

\[
0 \leq \frac{\sqrt{n + 1}}{n^2} \leq \frac{\sqrt{n + n}}{n^2} = \frac{\sqrt{2}}{n^{3/2}}
\]

for \( n \geq 1 \) and that \( \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^{3/2}} \) is a convergent \( p \)-series with \( p = \frac{3}{2} > 1 \). Therefore, the series \( \sum_{n=1}^{\infty} \frac{\sqrt{n + 1}}{n^2} \) converges by the Comparison Test and \( \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n + 1}}{n^2} \) is absolutely convergent.

(c) We use the Limit Comparison Test with \( \sum_{n=1}^{\infty} \frac{1}{n} \) which is a divergent \( p \)-series with \( p = 1 \leq 1 \).

\[
L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sqrt{n + 1}}{n} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + 1}} = 1
\]
Since $L = 1$ and $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.
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4. Determine whether the improper integrals converge or not (justify your answers):

(a) \[ \int_{1}^{+\infty} \frac{dx}{x^2 + x + 1} \]
(b) \[ \int_{0}^{\pi/2} \tan x \, dx \]

Solution:

(a) We will use the Comparison Test to show that the integral converges. Let \( g(x) = \frac{1}{x^2 + x + 1} \). We must choose a function \( f(x) \) that satisfies:

(1) \( \int_{1}^{+\infty} f(x) \, dx \) converges and (2) \( 0 \leq g(x) \leq f(x) \) for \( x \geq 1 \)

We choose \( f(x) = \frac{1}{x^2} \). This function satisfies the inequality:

\[
0 \leq g(x) \leq f(x) \\
0 \leq \frac{1}{x^2 + x + 1} \leq \frac{1}{x^2}
\]

for \( x \geq 1 \) because the denominator of \( g(x) \) is greater than the denominator of \( f(x) \) for these values of \( x \). Furthermore, the integral \( \int_{1}^{+\infty} f(x) \, dx = \int_{1}^{+\infty} \frac{1}{x^2} \, dx \) converges because it is a \( p \)-integral with \( p = 2 > 1 \). Therefore, the integral \( \int_{1}^{+\infty} g(x) \, dx = \int_{1}^{+\infty} \frac{1}{x^2 + x + 1} \, dx \) converges by the Comparison Test.

(b) We begin by noting that \( \tan x \) is undefined at \( x = \frac{\pi}{2} \). Thus, we replace the upper limit with \( R \) and take the limit as \( R \to \frac{\pi}{2} \).

\[
\int_{0}^{\pi/2} \tan x \, dx = \lim_{R \to \pi/2} \int_{0}^{R} \tan x \, dx \\
= \lim_{R \to \pi/2} \left[ - \ln |\cos x| \right]_{0}^{R} \\
= \lim_{R \to \pi/2} \left[ - \ln |\cos R| + \ln |\cos 0| \right] \\
= \infty + 0 \\
= \infty
\]

Therefore, the integral diverges.
5. Compute the arclength of the graph of \( y = (x + 1)^{3/2} + 1 \) between \( x = 0 \) and \( x = 2 \).

**Solution:** The arclength is:

\[
L = \int_a^b \sqrt{1 + (y')^2} \, dx
\]

\[
= \int_0^2 \sqrt{1 + \left( \frac{3}{2} (x + 1)^{1/2} \right)^2} \, dx
\]

\[
= \int_0^2 \sqrt{1 + \frac{9}{4} (x + 1)} \, dx
\]

\[
= \int_0^2 \sqrt{\frac{9}{4} x + \frac{13}{4}} \, dx
\]

We now use the \( u \)-substitution \( u = \frac{9}{4} x + \frac{13}{4} \). Then \( \frac{9}{4} \, du = dx \), the lower limit of integration changes from 0 to \( \frac{13}{4} \), and the upper limit of integration changes from 2 to \( \frac{31}{4} \).

\[
L = \int_0^2 \sqrt{\frac{9}{4} x + \frac{13}{4}} \, dx
\]

\[
= \frac{4}{9} \int_{3/4}^{31/4} \sqrt{u} \, du
\]

\[
= \frac{4}{9} \left[ \frac{2}{3} u^{3/2} \right]_{3/4}^{31/4}
\]

\[
= \frac{4}{9} \left[ \frac{2}{3} \left( \frac{31}{4} \right)^{3/2} - \frac{2}{3} \left( \frac{13}{4} \right)^{3/2} \right]
\]

\[
= \frac{8}{27} \left[ \left( \frac{31}{4} \right)^{3/2} - \left( \frac{13}{4} \right)^{3/2} \right]
\]
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6. Find the volume of the solid that is obtained by revolving the region below the graph 
   \( y = x^2 - 1 \) about the \( x \)-axis for \( 1 \leq x \leq 2 \).

**Solution:** We find the volume using the Disk method. The formula we will use is:

\[
V = \pi \int_{a}^{b} f(x)^2 \, dx
\]

where \( a = 1 \), \( b = 2 \), and \( f(x) = x^2 - 1 \). The volume is then:

\[
V = \pi \int_{1}^{2} f(x)^2 \, dx
\]

\[
= \pi \int_{1}^{2} (x^2 - 1)^2 \, dx
\]

\[
= \pi \int_{1}^{2} (x^4 - 2x^2 + 1) \, dx
\]

\[
= \pi \left[ \frac{x^5}{5} - \frac{2x^3}{3} + x \right]_{1}^{2}
\]

\[
= \pi \left[ \left( \frac{32}{5} - \frac{16}{3} + 2 \right) - \left( \frac{1}{5} - \frac{2}{3} + 1 \right) \right]
\]

\[
= \frac{38\pi}{15}
\]
7. Find the Maclaurin series around $x = 0$ for $f(x) = \ln(1 + 2x)$.

**Solution**: We begin by recalling the Maclaurin series for $\frac{1}{1-x}$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1.$$  

Upon replacing $x$ with $-2x$ we find that:

$$\frac{1}{1+2x} = 1 + (-2x) + (-2x)^2 + \cdots = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-2)^n x^n \text{ for } -\frac{1}{2} < x < \frac{1}{2}.$$  

Since

$$\int \frac{1}{1+2x} \, dx = \frac{1}{2} \ln(1+2x)$$

we have the relation

$$2 \int \frac{1}{1+2x} \, dx = \ln(1+2x)$$

provided that $x > -\frac{1}{2}$. The above relation yields the Maclaurin series for $\ln(1+2x)$ on the interval $-\frac{1}{2} < x < \frac{1}{2}$ as follows:

$$\ln(1+2x) = 2 \int \frac{1}{1+2x} \, dx = 2 \int \sum_{n=0}^{\infty} (-2)^n x^n \, dx = 2 \sum_{n=0}^{\infty} \frac{(-2)^n}{n+1} x^{n+1}.$$
8. Find the interval of convergence for \( \sum_{n=3}^{\infty} \frac{(2x)^n}{\ln n} \).

**Solution:** We use the Ratio Test to find the interval of convergence.

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(2x)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{(2x)^n} = \lim_{n \to \infty} \frac{\ln n}{\ln(n+1)} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{x^{n+1}}{x^n}
\]

\[
= 2|x| \lim_{n \to \infty} \frac{\ln n}{\ln(n+1)} = 2|x| \cdot (1) = 2|x|
\]

The series converges when \( \rho = 2|x| < 1 \) which gives us:

\[ |x| < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2} \]

We must now check the endpoints. Plugging \( x = \frac{1}{2} \) into the given power series we get:

\[
\sum_{n=1}^{\infty} \frac{(2(\frac{1}{2}))^n}{\ln n} = \sum_{n=1}^{\infty} \frac{1}{\ln n}
\]

This series diverges by a direct comparison with \( \sum_{n=1}^{\infty} \frac{1}{n} \) which is divergent.

Plugging in \( x = -\frac{1}{2} \) we get:

\[
\sum_{n=1}^{\infty} \frac{(2(-\frac{1}{2}))^n}{\ln n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}
\]

which converges by the Leibniz Test. Thus, the interval of convergence is:

\[ -\frac{1}{2} \leq x < \frac{1}{2} \]