1. Let \( P = (-1, 4, 1), \ Q = (1, 2, 9), \) and \( R = (5, 10, 1). \)

(a) Find the lengths of the sides \( PQ \) and \( PR \) of the triangle \( PQR. \)

(b) Find the interior angle at the vertex \( P \) of the triangle \( PQR. \)

(c) Find the equation of the plane containing \( P, \ Q, \) and \( R. \)

(d) Find the area of the triangle \( PQR. \)

Solution:

(a) The vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) are obtained by subtracting coordinates as follows:

\[
\overrightarrow{PQ} = \langle 1 - (-1), 2 - 4, 9 - 1 \rangle = \langle 2, -2, 8 \rangle
\]

\[
\overrightarrow{PR} = \langle 5 - (-1), 10 - 4, 1 - 1 \rangle = \langle 6, 6, 0 \rangle
\]

The lengths of sides \( PQ \) and \( PR \) are the magnitudes of the above vectors:

\[
\|\overrightarrow{PQ}\| = \sqrt{2^2 + (-2)^2 + 8^2} = \sqrt{72} = 6\sqrt{2}
\]

\[
\|\overrightarrow{PR}\| = \sqrt{6^2 + 6^2 + 0^2} = \sqrt{72} = 6\sqrt{2}
\]

(b) Use the dot product to determine the angle:

\[
\cos \theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\| \|\overrightarrow{PR}\|}
\]

\[
\cos \theta = \frac{(2)(6) + (-2)(6) + (8)(0)}{(6\sqrt{2})(6\sqrt{2})} = 0
\]

Therefore, the angle is \( \theta = \frac{\pi}{2}. \)
(c) A vector perpendicular to the plane is the cross product of $\vec{PQ}$ and $\vec{PR}$ which both lie in the plane.

$$\vec{n} = \vec{PQ} \times \vec{PR}$$

$$\vec{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 8 \\ 6 & 6 & 0 \end{vmatrix}$$

$$\vec{n} = \mathbf{i} \begin{vmatrix} -2 & 8 \\ 6 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 8 \\ 6 & 6 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -2 \\ 6 & 6 \end{vmatrix}$$

$$\vec{n} = \mathbf{i} [(-2)(0) - (8)(6)] - \mathbf{j} [(2)(0) - (8)(6)] + \mathbf{k} [(2)(6) - (-2)(6)]$$

$$\vec{n} = -48 \mathbf{i} + 48 \mathbf{j} + 24 \mathbf{k}$$

$$\vec{n} = \langle -48, 48, 24 \rangle$$

Using $P = (-1,4,1)$ as a point on the plane, we have:

$$-48(x + 1) + 48(y - 4) + 24(z - 1) = 0$$

(d) The area of the triangle is half the magnitude of the cross product of $\vec{PQ}$ and $\vec{PR}$, which represents the area of the parallelogram spanned by the two vectors:

$$A = \frac{1}{2} ||\vec{PQ} \times \vec{PR}||$$

$$A = \frac{1}{2} \sqrt{(-48)^2 + 48^2 + 24^2}$$

$$A = \frac{1}{2} (72)$$

$$A = 36$$
2. Consider a particle traveling along the curve $\vec{r}(t) = \langle t, 2e^t, e^{2t} \rangle$.

(a) Calculate the position, velocity, speed and acceleration of the particle at $t = 1$.

(b) What is the length of $\vec{r}(t)$ between $t = 1$ and $t = 2$?

(c) Find an equation for the tangent line to $\vec{r}(t)$ at $t = 0$.

(d) Calculate the curvature of $\vec{r}(t)$ at $t = 0$.

Solution:

(a) The velocity and acceleration are:

$$\vec{v}(t) = \vec{r}'(t) = \langle 1, 2e^t, 2e^{2t} \rangle$$

$$\vec{a}(t) = \vec{r}''(t) = \langle 0, 2e^t, 4e^{2t} \rangle$$

At $t = 1$ we have:

$$\vec{r}(1) = \langle 1, 2e, e^2 \rangle$$

$$\vec{v}(1) = \langle 1, 2e, 2e^2 \rangle$$

$$\vec{a}(1) = \langle 0, 2e, 4e^2 \rangle$$

$$v(1) = \sqrt{1^2 + (2e)^2 + (2e^2)^2}$$

$$= \sqrt{1 + 4e^2 + 4e^4}$$

(b) The length of the curve is:

$$L = \int_1^2 || \vec{r}'(t) || \ dt$$

$$L = \int_1^2 \sqrt{1 + 4e^{2t} + 4e^{4t}} \ dt$$

$$L = \int_1^2 \sqrt{(1 + 2e^{2t})^2} \ dt$$

$$L = \int_1^2 (1 + 2e^{2t}) \ dt$$

$$L = \left[ t + e^{2t} \right]_1^2$$

$$L = (2 + e^4) - (1 + e^2)$$

$$L = 1 + e^4 - e^2$$
(c) The tangent line equation is:

\[
\overrightarrow{L}(t) = \overrightarrow{r}(0) + t \overrightarrow{r}'(0)
\]

\[
\overrightarrow{L}(t) = \langle 0, 2, 1 \rangle + t \langle 1, 2, 2 \rangle
\]

(d) The curvature at \( t = 0 \) is:

\[
\kappa(0) = \frac{||\overrightarrow{r}'(0) \times \overrightarrow{r}''(0)||}{||\overrightarrow{r}'(0)||^3}
\]

\[
\kappa(0) = \frac{||\langle 1, 2, 2 \rangle \times \langle 0, 2, 4 \rangle||}{||\langle 1, 2, 2 \rangle||^3}
\]

\[
\kappa(0) = \frac{||\langle 1, 2, 2 \rangle||^3}{3^3}
\]

\[
\kappa(0) = \frac{6}{27}
\]

\[
\kappa(0) = \frac{2}{9}
\]
3. Evaluate the limits or determine they do not exist:

(a) \[ \lim_{(x,y) \to (3,4)} \frac{y - 3x}{x^2 + y^2} \]

(b) \[ \lim_{(x,y) \to (0,0)} \frac{2x^2 + y^2}{x^2 + y^2} \]

Solution:

(a) The function \( f(x, y) = \frac{y - 3x}{x^2 + y^2} \) is continuous at \((3, 4)\). Thus, we can evaluate the limit using substitution:

\[
\lim_{(x,y) \to (3,4)} \frac{y - 3x}{x^2 + y^2} = \frac{4 - 3(3)}{3^2 + 4^2} = \frac{-5}{25} = -\frac{1}{5}
\]

(b) We show that the limit does not exist by computing the limit of \( f(x, y) \) along two different paths that approach \((0, 0)\).

(i) For the first path we approach \((0, 0)\) along the \(x\)-axis from the right. In this case, we have \( y = 0 \) and \( x \to 0^+ \). The limit is then:

\[
\lim_{(x,y) \to (0,0)} \frac{2x^2 + y^2}{x^2 + y^2} = \lim_{x \to 0^+} \frac{2x^2 + 0^2}{x^2 + 0^2} = 2
\]

(ii) For the second path we approach \((0, 0)\) along the \(y\)-axis from above. In this case, we have \( x = 0 \) and \( y \to 0^+ \). The limit is then:

\[
\lim_{(x,y) \to (0,0)} \frac{2x^2 + y^2}{x^2 + y^2} = \lim_{y \to 0^+} \frac{2(0)^2 + y^2}{0^2 + y^2} = 1
\]

Since we get two different limits, the limit does not exist.