1. Let \( r(t) = \langle 4 \cos(2t), 5 \sin(2t), 3 \cos(2t) \rangle \).

(a) Find the velocity and acceleration of \( r(t) \), given as a function of \( t \).

(b) Find the principal unit normal vector when \( t = \pi \).

Solution:

(a) The velocity and acceleration vectors are the first and second derivatives of \( r(t) \), respectively.

\[
\begin{align*}
\mathbf{r}'(t) &= \langle -8 \sin(2t), 10 \cos(2t), -6 \sin(2t) \rangle, \\
\mathbf{r}''(t) &= \langle -16 \cos(2t), -20 \sin(2t), -12 \sin(2t) \rangle
\end{align*}
\]

(b) By definition, the principal unit normal vector is

\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}
\]

where

\[
\begin{align*}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \\
\mathbf{T}(t) &= \frac{\langle -8 \sin(2t), 10 \cos(2t), -6 \sin(2t) \rangle}{\sqrt{(-8 \sin(2t))^2 + (10 \cos(2t))^2 + (-6 \sin(2t))^2}} \\
\mathbf{T}(t) &= \frac{\langle -8 \sin(2t), 10 \cos(2t), -6 \sin(2t) \rangle}{\sqrt{64 \sin^2(2t) + 100 \cos^2(2t) + 36 \sin^2(2t)}} \\
\mathbf{T}(t) &= \frac{\langle -8 \sin(2t), 10 \cos(2t), -6 \sin(2t) \rangle}{\sqrt{100 \sin^2(2t) + 100 \cos^2(2t)}} \\
\mathbf{T}(t) &= \frac{\langle -8 \sin(2t), 10 \cos(2t), -6 \sin(2t) \rangle}{\sqrt{100}} \\
\mathbf{T}(t) &= \frac{\langle -8 \sin(2t), 10 \cos(2t), -6 \sin(2t) \rangle}{10} \\
\mathbf{T}(t) &= \langle -\frac{4}{5} \sin(2t), \cos(2t), -\frac{3}{5} \sin(2t) \rangle
\end{align*}
\]
is the unit tangent vector. Thus, the principal unit normal vector is

\[
N(t) = \frac{T'(t)}{|T'(t)|}
\]

\[
N(t) = \frac{\langle -\frac{8}{5} \cos(2t), -2 \sin(2t), -\frac{6}{5} \cos(2t) \rangle}{\sqrt{(-\frac{8}{5} \cos(2t))^2 + (-2 \sin(2t))^2 + (-\frac{6}{5} \cos(2t))^2}}
\]

\[
N(t) = \frac{\langle -\frac{8}{5} \cos(2t), -2 \sin(2t), -\frac{6}{5} \cos(2t) \rangle}{\sqrt{\frac{64}{25} \cos^2(2t) + 4 \sin^2(2t) + \frac{36}{25} \sin^2(2t)}}
\]

\[
N(t) = \frac{\langle -\frac{8}{5} \cos(2t), -2 \sin(2t), -\frac{6}{5} \cos(2t) \rangle}{\sqrt{4 \sin^2(2t) + 4 \cos^2(2t)}}
\]

\[
N(t) = \frac{\langle -\frac{8}{5} \cos(2t), -2 \sin(2t), -\frac{6}{5} \cos(2t) \rangle}{\sqrt{4}}
\]

\[
N(t) = \frac{\langle -\frac{8}{5} \cos(2t), -2 \sin(2t), -\frac{6}{5} \cos(2t) \rangle}{2}
\]

\[
N(t) = \langle -\frac{4}{5} \cos(2t), -\sin(2t), -\frac{3}{5} \cos(2t) \rangle
\]

When \( t = \pi \) we have

\[
N(\pi) = \langle -\frac{4}{5}, 0, -\frac{3}{5} \rangle
\]
2. Consider the curve \( r(t) = \langle \cos(t), \sin(t), t \rangle \).

(a) Graph the curve \( r(t) \) for \( 0 \leq t \leq 2\pi \). Indicate in your graph the endpoints and the direction as \( t \) increases.

(b) Find the speed of \( r(t) \) when \( t = 0 \) and the unit tangent vector \( T(t) \) when \( t = \frac{\pi}{2} \).

Solution:

(a) A plot of the curve is shown below:

The endpoints of the curve are (1, 0, 0) and (1, 0, 2\( \pi \)). The direction is counterclockwise as viewed from above.

(b) By definition, the speed is \( v(t) = ||r'(t)|| = ||\langle -\sin(t), \cos(t), 1 \rangle|| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2} \). At \( t = 0 \) we have

\[
v(0) = \sqrt{2}
\]

since the speed is constant. By definition, the unit tangent vector is \( T(t) = r'(t) / ||r'(t)|| = \frac{1}{\sqrt{2}} (-\sin(t), \cos(t), 1) \). Thus, at \( t = \frac{\pi}{2} \) we have

\[
T(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} (-1, 0, 1)
\]
3. Let \( r_1(t) = \langle t^2, t^2 - 2t, t + 2 \rangle \) and \( r_2(s) = \langle s, -1, 2s + 1 \rangle \).

(a) Find the point or points, if any, at which the curves \( r_1(t) \) and \( r_2(s) \) intersect.

(b) Find the area of the parallelogram spanned by the two vectors \( r_1'(0) \) and \( r_1'(2) \).

Solution:

(a) The curves will intersect if there exist numbers \( t \) and \( s \) such that \( r_1(t) = r_2(s) \). This will occur if there is a solution to the system of equations:

\[
t^2 = s, \quad t^2 - 2t = -1, \quad t + 2 = 2s + 1
\]

The second equation leads to \( t^2 - 2t + 1 = 0 \) and, thus, \( t = 1 \). Plugging this into the first and third equations gives \( s = 1 \) in both cases. Therefore, the point of intersection is

\[
r_1(1) = \langle 1, -1, 3 \rangle
\]

(b) The derivative of \( r_1(t) \) is \( r_1'(t) = \langle 2t, 2t - 2, 1 \rangle \). The parallelogram is then spanned by

\[
u = r_1'(0) = \langle 0, -2, 1 \rangle \quad \text{and} \quad v = r_1'(2) = \langle 4, 2, 1 \rangle
\]

The area of this parallelogram is:

\[
A = ||u \times v||
\]

\[
A = ||\langle 0, -2, 1 \rangle \times \langle 4, 2, 1 \rangle||
\]

\[
A = ||\langle -4, 4, 8 \rangle||
\]

\[
A = \sqrt{(-4)^2 + 4^2 + 8^2}
\]

\[
A = 4\sqrt{6}
\]
4. Find the equation of the line through the point \( P = (1, -3, 2) \) that is perpendicular to the vectors \( \langle 1, 0, 2 \rangle \) and \( \langle 2, 1, 0 \rangle \).

Solution: The vector equation for a line containing the point \( P_0(x_0, y_0, z_0) \) and parallel to the vector \( \mathbf{v} = \langle a, b, c \rangle \) is

\[
\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle
\]

The vector \( \mathbf{v} \) is the cross product of \( \langle 1, 0, 2 \rangle \) and \( \langle 2, 1, 0 \rangle \) since \( \mathbf{v} \) will be perpendicular to both vectors.

\[
\mathbf{v} = \langle 1, 0, 2 \rangle \times \langle 2, 1, 0 \rangle = \langle -2, 4, 1 \rangle
\]

Therefore, the equation for the line is

\[
\mathbf{r}(t) = \langle 1, -3, 2 \rangle + t \langle -2, 4, 1 \rangle
\]
5. Show that the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2 + y^2}$ does not exist.

**Solution**: We use the two-path test to show that the limit does not exist. Let the first path be the line $y = 0$ as $x \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2 + y^2} = \lim_{x \to 0^+} \frac{x \cdot 0}{3x^2 + 0^2} = 0$$

Now let the second path be the line $y = x$ as $x \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2 + y^2} = \lim_{x \to 0^+} \frac{x \cdot x}{3x^2 + x^2} = \lim_{x \to 0^+} \frac{x^2}{4x^2} = \frac{1}{4}$$

Since the limits are different along different paths, the limit does not exist.
6. Find the length of the curve \( \mathbf{r}(t) = (2 \cos(3t), 3t, 2 \sin(3t)) \) between \((2, 0, 0)\) and \((2, 2\pi, 0)\).

**Solution:** The length of the curve is computed using the formula

\[
L = \int_{a}^{b} ||\mathbf{r}'(t)|| \, dt
\]

The derivative \( \mathbf{r}'(t) \) and its magnitude are:

\[
||\mathbf{r}'(t)|| = ||\langle -6 \sin(3t), 3, 6 \cos(3t) \rangle||
\]

\[
||\mathbf{r}'(t)|| = \sqrt{36 \sin^2(3t) + 9 + 36 \cos^2(3t)}
\]

\[
||\mathbf{r}'(t)|| = \sqrt{45}
\]

The endpoints of the curve correspond to \( t = 0 \) and \( t = \frac{2\pi}{3} \), respectively. Therefore, the length is

\[
L = \int_{0}^{2\pi/3} \sqrt{45} \, dt = \frac{2\pi \sqrt{45}}{3} = 2\pi \sqrt{5}
\]