Math 210, Exam 1, Practice Fall 2009

Problem 1 Solution

1. Let \( A = (1, -1, 2), \ B = (0, -1, 1), \ C = (2, 1, 1). \)

(a) Find the vector equation of the plane through \( A, B, C. \)

(b) Find the area of the triangle with these three vertices.

Solution:

(a) In order to find the vector equation of the plane we need a point that lies in the plane and a vector \( \vec{n} \) perpendicular to it. We let \( \vec{n} \) be the cross product of \( \vec{AB} = \langle -1, 0, -1 \rangle \) and \( \vec{BC} = \langle 2, 2, 0 \rangle \) because these vectors lie in the plane.

\[
\vec{n} = \vec{AB} \times \vec{BC}
\]

\[
\vec{n} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-1 & 0 & -1 \\
2 & 2 & 0
\end{vmatrix}
\]

\[
\vec{n} = \hat{i} \left\{ 0 \cdot -1 - (-1) \cdot 0 \right\} - \hat{j} \left\{ -1 \cdot 0 - (-1) \cdot 2 \right\} + \hat{k} \left\{ -1 \cdot 2 - 0 \cdot 2 \right\}
\]

\[
\vec{n} = 2\hat{i} - 2\hat{j} - 2\hat{k}
\]

\[
\vec{n} = \langle 2, -2, -2 \rangle
\]

Using \( A = (1, -1, 2) \) as a point in the plane, we have:

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
0 & -1 & -2 \\
2 & 2 & 0
\end{vmatrix}
\cdot \langle 2, -2, -2 \rangle = 0
\]

as the vector equation for the plane containing \( A, B, C. \)

(b) The area of the triangle is half the magnitude of the cross product of \( \vec{AB} \) and \( \vec{BC}, \) which represents the area of the parallelogram spanned by the two vectors:

\[
A = \frac{1}{2} \left| \vec{AB} \times \vec{BC} \right|
\]

\[
A = \frac{1}{2} \sqrt{2^2 + (-2)^2 + (-2)^2}
\]

\[
A = \frac{1}{2} \sqrt{12}
\]

\[
A = \sqrt{3}
\]
2. Find the vector of length one in the direction of $\mathbf{v} - \mathbf{u}$ where $\mathbf{v} = \langle 7, 5, 3 \rangle$ and $\mathbf{u} = \langle 4, 5, 7 \rangle$.

Solution: First, the vector $\mathbf{v} - \mathbf{u}$ is:

$$\mathbf{v} - \mathbf{u} = \langle 7, 5, 3 \rangle - \langle 4, 5, 7 \rangle = \langle 3, 0, -4 \rangle$$

Next, we convert this vector into a unit vector by multiplying by the reciprocal of its magnitude.

$$\hat{e} = \frac{1}{||\mathbf{v} - \mathbf{u}||} (\mathbf{v} - \mathbf{u})$$

$$\hat{e} = \frac{1}{\sqrt{3^2 + 0^2 + (-4)^2}} \langle 3, 0, -4 \rangle$$

$$\hat{e} = \frac{1}{5} \langle 3, 0, -4 \rangle$$

$$\hat{e} = \langle \frac{3}{5}, 0, -\frac{4}{5} \rangle$$
3. Let \( \mathbf{r}(t) = \langle 3t - 1, e^t, \cos(t) \rangle \).

(a) Find the unit tangent vector \( \mathbf{T} \) to the path \( \mathbf{r}(t) \) at \( t = 0 \).

(b) Find the speed, \( ||\mathbf{r}'(t)|| \) at \( t = 0 \).

**Solution:**

(a) The derivative of \( \mathbf{r}(t) \) is \( \mathbf{r}'(t) = \langle 3, e^t, -\sin t \rangle \). At \( t = 0 \) we have \( \mathbf{r}'(0) = \langle 3, 1, 0 \rangle \).

The unit tangent vector at \( t = 0 \) is then:

\[
\mathbf{T}(0) = \frac{1}{\|\mathbf{r}'(0)\|} \mathbf{r}'(0) = \frac{1}{\sqrt{3^2 + 1^2 + 0^2}} \langle 3, 1, 0 \rangle = \frac{1}{\sqrt{10}} \langle 3, 1, 0 \rangle = \langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \rangle
\]

(b) The speed at \( t = 0 \) is \( \|\mathbf{r}'(0)\| = \sqrt{10} \).
4. Given a point \( P = (0, 1, 2) \) and the vectors \( \vec{u} = (1, 0, 1) \) and \( \vec{v} = (2, 3, 0) \), find

(a) an equation for the plane that contains \( P \) and whose normal vector is perpendicular to the two vectors \( \vec{u} \) and \( \vec{v} \),

(b) a set of parametric equations of the line through \( P \) and in the direction of \( \vec{v} \).

Solution:

(a) In order to find an equation for the plane we need a point that lies in the plane and a vector \( \vec{n} \) perpendicular to it. We let \( \vec{n} \) be the cross product of \( \vec{u} \) and \( \vec{v} \).

\[
\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & \hat{k} \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -3\hat{i} + 2\hat{j} + 3\hat{k}
\]

Using \( P = (0, 1, 2) \) as a point in the plane, we have:

\[
-3(x - 0) + 2(y - 1) + 3(z - 2) = 0
\]

as the equation for the plane.

(b) A set of parametric equations of the line through \( P = (0, 1, 2) \) and in the direction of \( \vec{v} = (2, 3, 0) \) is:

\[
x = 0 + 2t, \quad y = 1 + 3t, \quad z = 2 + 0t
\]
5. Find the speed and arclength of the path \( \vec{r}(t) = (3 \cos t, 4 \cos t, 5 \sin t) \) where \( 0 \leq t \leq 2 \).

**Solution:** The derivative of \( \vec{r}(t) \) is \( \vec{r}'(t) = (-3 \sin t, -4 \sin t, 5 \cos t) \). Speed is the magnitude of \( \vec{r}'(t) \).

\[
\| \vec{r}'(t) \| = \sqrt{(-3 \sin t)^2 + (-4 \sin t)^2 + (5 \cos t)^2} \\
\| \vec{r}'(t) \| = \sqrt{9 \sin^2 t + 16 \sin^2 t + 25 \cos^2 t} \\
\| \vec{r}'(t) \| = \sqrt{25 \sin^2 t + 25 \cos^2 t} \\
\| \vec{r}'(t) \| = \sqrt{25} \\
\| \vec{r}'(t) \| = 5
\]

The arclength of the path is then:

\[
L = \int_0^2 \| \vec{r}'(t) \| \, dt \\
L = \int_0^2 5 \, dt \\
L = 5t \bigg|_0^2 \\
L = 10
\]
6. Find the curvature at \( t = 0 \) for the curve \( \mathbf{r}(t) = e^t \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k} \).

**Solution:** The curvature formula we will use is:

\[
\kappa(0) = \frac{||\mathbf{r}''(0) \times \mathbf{r}'(0)||}{||\mathbf{r}'(0)||^3}
\]

The first two derivatives of \( \mathbf{r}(t) = \langle e^t, t^2, t \rangle \) are:

\[
\mathbf{r}'(t) = \langle e^t, 2t, 1 \rangle
\]

\[
\mathbf{r}''(t) = \langle e^t, 2, 0 \rangle
\]

We now evaluate the derivatives at \( t = 0 \).

\[
\mathbf{r}'(0) = \langle e^0, 2(0), 1 \rangle = \langle 1, 0, 1 \rangle
\]

\[
\mathbf{r}''(0) = \langle e^0, 2, 0 \rangle = \langle 1, 2, 0 \rangle
\]

The cross product of these vectors is:

\[
\mathbf{r}'(0) \times \mathbf{r}''(0) = \mathbf{i} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{vmatrix}
\]

\[
\mathbf{r}'(0) \times \mathbf{r}''(0) = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}
\]

We can now compute the curvature.

\[
\kappa(0) = \frac{||\mathbf{r}'(0) \times \mathbf{r}''(0)||}{||\mathbf{r}'(0)||^3}
\]

\[
\kappa(0) = \frac{\sqrt{(-2)^2 + 1^2 + 2^2}}{(\sqrt{1^2 + 0^2 + 1^2})^3}
\]

\[
\kappa(0) = \frac{\sqrt{9}}{2\sqrt{2}}
\]

\[
\kappa(0) = \frac{3}{2\sqrt{2}}
\]
7. Let $\vec{r}(t) = \langle t, \cos t, \sin t \rangle$.

(a) Find the velocity vector, $\vec{r}'(t)$.

(b) Find the acceleration vector, $\vec{r}''(t)$.

(c) Find the component of acceleration in the direction of the velocity when $t = 0$.

Solution:

(a) The velocity vector is $\vec{v}(t) = \vec{r}'(t) = \langle 1, -\sin t, \cos t \rangle$.

(b) The acceleration vector is $\vec{a}(t) = \vec{r}''(t) = \langle 0, -\cos t, -\sin t \rangle$.

(c) At $t = 0$, the velocity and acceleration vectors are:

$\vec{v}(0) = \langle 1, -\sin 0, \cos 0 \rangle = \langle 1, 0, 1 \rangle$

$\vec{a}(0) = \langle 0, -\cos 0, -\sin 0 \rangle = \langle 0, -1, 0 \rangle$

The acceleration can be decomposed into tangential and normal components.

$\vec{a} = a_T \vec{T} + a_N \vec{N}$

By definition, the component of acceleration in the direction of the velocity is $a_T$. The formula and subsequent computation are shown below.

$$a_T = \frac{\vec{a}(0) \cdot \vec{v}(0)}{||\vec{v}(0)||}$$

$$a_T = \frac{\langle 0, -1, 0 \rangle \cdot \langle 1, 0, 1 \rangle}{||\langle 1, 0, 1 \rangle||}$$

$$a_T = \frac{(0)(1) + (-1)(0) + (0)(1)}{\sqrt{1^2 + 0^2 + 1^2}}$$

$$a_T = 0$$
8. Let \( f(x, y) = \frac{1}{2}x^2 - y \). Sketch the three level curves on which \( f(x, y) = -1 \) or 0 or 1 in the square \(-2 \leq x \leq 2, -2 \leq y \leq 2\).

**Solution:** The level curves of \( f(x, y) = \frac{1}{2}x^2 - y \) are the curves obtained by setting \( f(x, y) \) to a constant \( C \).

\[
C = \frac{1}{2}x^2 - y \iff y = \frac{1}{2}x^2 - C
\]

These curves are parabolas with vertices at \((0, -C)\) and are sketched below for the values \( C = -1, 0, 1 \).
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Problem 9 Solution

9. Find the partial derivatives

\[ \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} \]

for the function \( f(x, y) = 2x + 3xy - 5y^2 \).

Solution: The first partial derivatives of \( f(x, y) \) are

\[ \frac{\partial f}{\partial x} = 2 + 3y \]
\[ \frac{\partial f}{\partial y} = 3x - 10y \]

The second mixed partial derivative \( \frac{\partial^2 f}{\partial x \partial y} \) is

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (3x - 10y) = 3 \]
10. Find the partial derivatives \( \frac{\partial^2 f}{\partial x^2} \) and \( \frac{\partial^2 f}{\partial y^2} \)
for the function \( f(x, y) = e^{2x} \cos(2y) \).

**Solution:** The first partial derivatives of \( f(x, y) \) are

\[
\frac{\partial f}{\partial x} = 2e^{2x} \cos(2y) \\
\frac{\partial f}{\partial y} = -2e^{2x} \sin(2y)
\]

The second partial derivative \( \frac{\partial^2 f}{\partial x^2} \) is

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = 4e^{2x} \cos(2y)
\]

The second partial derivative \( \frac{\partial^2 f}{\partial y^2} \) is

\[
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = -4e^{2x} \cos(2y)
\]