1. Let \( f(x, y, z) = \sin(xy - 8) - \ln(z + 1) + \frac{2x}{y - z} \).

(a) Compute the gradient \( \nabla f \) as a function of \( x, y, \) and \( z \).

(b) Find the equation of the tangent plane to the surface \( f(x, y, z) = 4 \) at \( (4, 2, 0) \).

(c) Compute the directional derivative \( D_{\vec{u}} f(4, 2, 0) \) where \( \vec{u} \) is a unit vector in the direction of \( \langle -2, 1, 0 \rangle \).

Solution:

(a) By definition, the gradient of \( f(x, y, z) \) is:

\[
\nabla f = \langle f_x, f_y, f_z \rangle
\]

The partial derivatives of \( f \) are:

\[
\begin{align*}
f_x &= y \cos(xy - 8) + \frac{2}{y - z} \\
f_y &= x \cos(xy - 8) - \frac{2x}{(y - z)^2} \\
f_z &= -\frac{1}{z + 1} + \frac{2x}{(y - z)^2}
\end{align*}
\]

Thus, the gradient is:

\[
\nabla f = \left( y \cos(xy - 8) + \frac{2}{y - z}, x \cos(xy - 8) - \frac{2x}{(y - z)^2}, -\frac{1}{z + 1} + \frac{2x}{(y - z)^2} \right)
\]

(b) We use the following formula for the equation for the tangent plane:

\[
f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0
\]

because the surface equation is given in implicit form. Note that \( \vec{n} = \nabla f(a, b, c) = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle \) is a vector normal to the surface \( f(x, y, z) = C \) and, thus, to the tangent plane at the point \( (a, b, c) \) on the surface.

The partial derivatives evaluated at \( (4, 2, 0) \) are:

\[
\begin{align*}
f_x(4, 2, 0) &= 2 \cos((4)(2) - 8) + \frac{2}{2 - 0} = 3 \\
f_y(4, 2, 0) &= 4 \cos((4)(2) - 8) - \frac{2(4)}{(2 - 0)^2} = 2 \\
f_z(4, 2, 0) &= -\frac{1}{0 + 1} + \frac{2(4)}{(2 - 0)^2} = 1
\end{align*}
\]
Thus, the tangent plane equation is:

$$3(x - 4) + 2(y - 2) + (z - 0) = 0$$

(c) By definition, the directional derivative of $f(x, y, z)$ at $(4, 2, 0)$ in the direction of $\hat{u}$ is:

$$D_{\hat{u}}f(4, 2, 0) = \nabla f(4, 2, 0) \cdot \hat{u}$$

From part (b), we have $\nabla f(4, 2, 0) = \langle 3, 2, 1 \rangle$. Recalling that $\hat{u}$ must be a unit vector, we multiply $\langle -2, 1, 0 \rangle$ by the reciprocal of its magnitude.

$$\hat{u} = \frac{1}{|\langle -2, 1, 0 \rangle|} \langle -2, 1, 0 \rangle = \frac{1}{\sqrt{5}} \langle -2, 1, 0 \rangle$$

Therefore, the directional derivative is:

$$D_{\hat{u}}f(4, 2, 0) = \nabla f(4, 2, 0) \cdot \hat{u}$$

$$D_{\hat{u}}f(4, 2, 0) = \langle 3, 2, 1 \rangle \cdot \frac{1}{\sqrt{5}} \langle -2, 1, 0 \rangle$$

$$D_{\hat{u}}f(4, 2, 0) = \frac{1}{\sqrt{5}} [(3)(-2) + (2)(1) + (1)(0)]$$

$$D_{\hat{u}}f(4, 2, 0) = -\frac{4}{\sqrt{5}}$$
2. Let \( f(x, y) = x^2 + y^2 - y \), and let \( D \) be the bounded region defined by the inequalities \( y \geq 0 \) and \( y \leq 1 - x^2 \).

(a) Find and classify the critical points of \( f(x, y) \).

(b) Sketch the region \( D \).

(c) Find the absolute maximum and minimum values of \( f \) on the region \( D \), and list the points where these values occur.

**Solution:** First we note that the domain of \( f(x, y) \) is bounded and closed, i.e. compact, and that \( f(x, y) \) is continuous on the domain. Thus, we are guaranteed to have absolute extrema.

(a) The partial derivatives of \( f \) are \( f_x = 2x \) and \( f_y = 2y - 1 \). The critical points of \( f \) are all solutions to the system of equations:

\[
\begin{align*}
    f_x &= 2x = 0 \\
    f_y &= 2y - 1 = 0
\end{align*}
\]

The only solution is \( x = 0 \) and \( y = \frac{1}{2} \), which is an interior point of \( D \). The function value at the critical point is:

\[
    f(0, \frac{1}{2}) = -\frac{1}{4}
\]

(b) The region \( D \) (shaded) is plotted below along with level curves of \( f(x, y) \).
(c) We must now determine the minimum and maximum values of $f$ on the boundary of $D$. To do this, we must consider each part of the boundary separately:

**Part I**: Let this part be the line segment between $(-1, 0)$ and $(1, 0)$. On this part we have $y = 0$ and $-1 \leq x \leq 1$. We now use the fact that $y = 0$ to rewrite $f(x, y)$ as a function of one variable that we call $g_I(x)$.

$$f(x, y) = x^2 + y^2 - y$$
$$g_I(x) = x^2 + 0^2 - 0$$
$$g_I(x) = x^2$$

The critical points of $g_I(x)$ are:

$$g_I'(x) = 0$$
$$2x = 0$$
$$x = 0$$

Evaluating $g_I(x)$ at the critical point $x = 0$ and at the endpoints of the interval $-1 \leq x \leq 1$, we find that:

$$g_I(0) = 0, \quad g_I(-1) = 1, \quad g_I(1) = 1$$

Note that these correspond to the function values:

$$f(0, 0) = 0, \quad f(-1, 0) = 1, \quad f(1, 0) = 1$$

**Part II**: Let this part be the parabola $y = 1 - x^2$ on the interval $-1 \leq x \leq 1$. We now use the fact that $y = 1 - x^2$ to rewrite $f(x, y)$ as a function of one variable that we call $g_{II}(x)$.

$$f(x, y) = x^2 + y^2 - y$$
$$g_{II}(x) = x^2 + (1 - x^2)^2 - (1 - x^2)$$
$$g_{II}(x) = x^2 + 1 - 2x^2 + x^4 - 1 + x^2$$
$$g_{II}(x) = x^4$$

The critical points of $g_{II}(x)$ are:

$$g_{II}'(x) = 0$$
$$4x^3 = 0$$
$$x = 0$$

Evaluating $g_{II}(x)$ at the critical point $x = 0$ and at the endpoints of the interval $-1 \leq x \leq 1$, we find that:

$$g_{II}(0) = 0, \quad g_{II}(-1) = 1, \quad g_{II}(1) = 1$$

Note that these correspond to the function values:

$$f(0, 1) = 0, \quad f(-1, 0) = 1, \quad f(1, 0) = 1$$
Finally, after comparing these values of $f$ we find that the absolute maximum of $f$ is 1 at the points $(-1,0)$ and $(1,0)$ and that the absolute minimum of $f$ is $-\frac{1}{4}$ at the point $(0,\frac{1}{2})$.

**Note:** In the figure from part (b) we see that the level curves of $f$ are circles centered at $(0,\frac{1}{2})$. It is clear that the absolute minimum of $f$ occurs at $(0,\frac{1}{2})$ and that the absolute maximum of $f$ occurs at $(-1,0)$ and $(1,0)$, which are points on the largest circle centered at $(0,\frac{1}{2})$ that contains points in $D$. 
Consider the iterated integral \( \int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos (y^2) \, dy \, dx \).

(a) Sketch the region of integration.

(b) Compute the integral. (Hint: First reverse the order of integration.)

Solution:

(a) The region of integration \( \mathcal{R} \) is sketched below:

(b) The region \( \mathcal{R} \) can be described as follows:

\[
\mathcal{R} = \{(x, y) : 0 \leq x \leq y, \ 0 \leq y \leq \sqrt{\pi}\}
\]

Therefore, the value of the integral is:

\[
\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos (y^2) \, dy \, dx = \int_0^{\sqrt{\pi}} \int_0^y \cos (y^2) \, dx \, dy
\]

\[
= \int_0^{\sqrt{\pi}} \cos (y^2) \left[ x \right]_0^y
\]

\[
= \int_0^{\sqrt{\pi}} y \cos (y^2) \, dy
\]

\[
= \left[ \frac{1}{2} \sin (y^2) \right]_0^{\sqrt{\pi}}
\]

\[
= \frac{1}{2} \sin (\sqrt{\pi}^2) - \frac{1}{2} \sin (0^2)
\]

\[
= 0
\]
4. Let $Q$ be the part of the unit disk that lies in the second quadrant, i.e.

$$Q = \{(x, y) \mid x \leq 0, \; y \geq 0, \; x^2 + y^2 \leq 1\}$$

(a) Write an iterated integral in polar coordinates that represents the area of $Q$ and compute this area.

(b) Compute $\iint_{Q} (3x^2 + 3y^2) \, dA$.

(c) Compute the average value of $f(x, y) = x^2 + y^2$ over $Q$.

Solution:

(a) The region $Q$ can be described in polar coordinates as:

$$Q = \{(r, \theta) \mid 0 \leq r \leq 1, \; \frac{\pi}{2} \leq \theta \leq \pi\}$$

Using the fact that $dA = r \, dr \, d\theta$ in polar coordinates, the area of $Q$ is:

$$\text{Area}(Q) = \iint_{Q} 1 \, dA$$

$$= \int_{\pi/2}^{\pi} \int_{0}^{1} r \, dr \, d\theta$$

$$= \int_{\pi/2}^{\pi} \left[ \frac{1}{2} r^2 \right]_{0}^{1} \, d\theta$$

$$= \int_{\pi/2}^{\pi} \frac{1}{2} \, d\theta$$

$$= \left[ \frac{1}{2} \theta \right]_{\pi/2}^{\pi}$$

$$= \frac{\pi}{4}$$

(b) The function $f(x, y) = 3x^2 + y^2$ can be written in polar coordinates as:

$$f(r, \theta) = 3r^2$$
The integral of $f(r, \theta)$ over the region $Q$ is then:

\[
\text{Area}(Q) = \int\int_{Q} f(r, \theta) \, dA \\
= \int_{\pi/2}^{\pi} \int_{0}^{1} 3r^2 \cdot r \, dr \, d\theta \\
= \int_{\pi/2}^{\pi} \left[ \frac{3}{4}r^4 \right]_{0}^{1} \, d\theta \\
= \int_{\pi/2}^{\pi} \frac{3}{4} \, d\theta \\
= \left[ \frac{3}{4}\theta \right]_{\pi/2}^{\pi} \\
= \frac{3\pi}{8}
\]

(c) We use the following formula to compute the average value of $f$:

\[
\bar{f} = \frac{\iint_{A} f(x, y) \, dA}{\iint_{A} 1 \, dA}
\]

The function $f(x, y) = x^2 + y^2$ written in polar coordinates is:

\[
f(r, \theta) = r^2
\]

The integral of $f(r, \theta)$ over the region $Q$ is then:

\[
\iint_{Q} f(x, y) \, dA = \int_{\pi/2}^{\pi} \int_{0}^{1} r^2 \cdot r \, dr \, d\theta \\
= \frac{1}{3} \int_{\pi/2}^{\pi} \int_{0}^{1} 3r^2 \cdot r \, dr \, d\theta \\
= \frac{1}{3} \cdot \frac{3\pi}{8} \\
= \frac{\pi}{8}
\]

where we used the result from part (b). The integral $\iint_{A} 1 \, dA = \frac{\pi}{4}$ was computed in part (a). Thus, the average value of $f$ is:

\[
\bar{f} = \frac{\pi}{8} \cdot \frac{1}{\frac{\pi}{4}} = \frac{1}{2}
\]