Problem 1 Solution

1. Let \( f(x, y) = \frac{1}{3}x^3 + y^2 - xy \). Find all critical points of \( f(x, y) \) and classify each as a local maximum, local minimum, or saddle point.

**Solution:** By definition, an interior point \((a, b)\) in the domain of \( f \) is a **critical point** of \( f \) if either

1. \( f_x(a, b) = f_y(a, b) = 0 \), or
2. one (or both) of \( f_x \) or \( f_y \) does not exist at \((a, b)\).

The partial derivatives of \( f(x, y) = \frac{1}{3}x^3 + y^2 - xy \) are \( f_x = x^2 - y \) and \( f_y = 2y - x \). These derivatives exist for all \((x, y)\) in \( \mathbb{R}^2 \). Thus, the critical points of \( f \) are the solutions to the system of equations:

\[
\begin{align*}
    f_x &= x^2 - y = 0 \\
    f_y &= 2y - x = 0
\end{align*}
\]

Solving Equation (1) for \( y \) we get:

\( y = x^2 \) (3)

Substituting this into Equation (2) and solving for \( x \) we get:

\[
\begin{align*}
    2y - x &= 0 \\
    2(x^2) - x &= 0 \\
    x(2x - 1) &= 0
\end{align*}
\]

\( \iff x = 0 \quad \text{or} \quad x = \frac{1}{2} \)

We find the corresponding \( y \)-values using Equation (3): \( y = x^2 \).

- If \( x = 0 \), then \( y = 0^2 = 0 \).
- If \( x = \frac{1}{2} \), then \( y = (\frac{1}{2})^2 = \frac{1}{4} \).

Thus, the critical points are \((0, 0)\) and \((\frac{1}{2}, \frac{1}{4})\).

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of \( f \) are:

\( f_{xx} = 2x, \quad f_{yy} = 2, \quad f_{xy} = -1 \)

The discriminant function \( D(x, y) \) is then:

\[
\begin{align*}
    D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\
    D(x, y) &= (2x)(2) - (-1)^2 \\
    D(x, y) &= 4x - 1
\end{align*}
\]

The values of \( D(x, y) \) at the critical points and the conclusions of the Second Derivative Test are shown in the table below.
<table>
<thead>
<tr>
<th>$(a, b)$</th>
<th>$D(a,b)$</th>
<th>$f_{xx}(a,b)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>$-1$</td>
<td>$0$</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>$(\frac{1}{2}, \frac{1}{4})$</td>
<td>$1$</td>
<td>$1$</td>
<td>Local Minimum</td>
</tr>
</tbody>
</table>

Recall that $(a,b)$ is a saddle point if $D(a,b) < 0$ and that $(a,b)$ corresponds to a local minimum of $f$ if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$.

Figure 1: Pictured above are level curves of $f(x,y)$. Darker colors correspond to smaller values of $f(x,y)$. It is apparent that $(0,0)$ is a saddle point and $(\frac{1}{2}, \frac{1}{4})$ corresponds to a local minimum.
2. Find the minimum and maximum of the function \( f(x, y, z) = x - y - z \) on the ellipsoid \( R = \{(x, y, z) \mid \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} = 1 \} \).

**Solution:** We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that \( R \) is compact which guarantees the existence of absolute extrema of \( f \). Then, let \( g(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} = 1 \). We look for solutions to the following system of equations:

\[
\begin{align*}
f_x &= \lambda g_x, & f_y &= \lambda g_y, & f_z &= \lambda g_z, \quad g(x, y, z) &= 1
\end{align*}
\]

which, when applied to our functions \( f \) and \( g \), give us:

\[
\begin{align*}
1 &= \lambda \left( \frac{2x}{4} \right) \\
-1 &= \lambda \left( \frac{2y}{9} \right) \\
-1 &= \lambda \left( \frac{2z}{3} \right)
\end{align*}
\]

\[
\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} = 1
\]

To solve the system of equations, we first solve Equations (1)-(3) for the variables \( x, y, \) and \( z \) in terms of \( \lambda \) to get:

\[
x = \frac{4}{2\lambda}, \quad y = -\frac{9}{2\lambda}, \quad z = -\frac{3}{2\lambda}
\]

We then plug Equations (5) into Equation (4) and simplify.

\[
\begin{align*}
\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} &= 1 \\
\left( \frac{2}{2\lambda} \right)^2 + \left( \frac{9}{2\lambda} \right)^2 + \left( \frac{3}{2\lambda} \right)^2 &= 1 \\
\frac{16}{4\lambda^2} + \frac{81}{4\lambda^2} + \frac{9}{4\lambda^2} &= 1 \\
\frac{16}{4} + \frac{81}{9} + \frac{9}{3} &= 1
\end{align*}
\]

At this point we multiply both sides of the equation by \( 4\lambda^2 \) to get:

\[
4\lambda^2 \left( \frac{16}{4\lambda^2} + \frac{81}{4\lambda^2} + \frac{9}{4\lambda^2} \right) = 4\lambda^2(1)
\]

\[
\frac{16}{4} + \frac{81}{9} + \frac{9}{3} = 4\lambda^2 \\
4 + 9 + 3 = 4\lambda^2 \\
\lambda^2 = 4 \\
\lambda = \pm 2
\]
• When \( \lambda = 2 \), Equations (5) give us the first candidate for the location of an extreme value:

\[
x = 1, \quad y = -\frac{9}{4}, \quad z = -\frac{3}{4}
\]

• When \( \lambda = -2 \), Equations (5) give us the second candidate for the location of an extreme value:

\[
x = -1, \quad y = \frac{9}{4}, \quad z = \frac{3}{4}
\]

Evaluating \( f(x, y, z) \) at these points we find that:

\[
f \left(1, -\frac{9}{4}, -\frac{3}{4} \right) = 1 - \left(-\frac{9}{4}\right) - \left(-\frac{3}{4}\right) = 4
\]

\[
f \left(-1, \frac{9}{4}, \frac{3}{4}\right) = -1 - \left(\frac{9}{4}\right) - \left(\frac{3}{4}\right) = -4
\]

Therefore, the absolute maximum value of \( f \) on \( R \) is 4 and the absolute minimum of \( f \) on \( R \) is \(-4\).

**Note:** The level surfaces \( f(x, y, z) = 4 \) and \( f(x, y, z) = -4 \) are planes tangent to the ellipsoid at the critical points.
3. Consider the double integral: \( \int_0^4 \int_0^{y^2} \frac{x^3}{4 - \sqrt{x}} \, dx \, dy \).

(a) Sketch the region of integration.

(b) Change the order of integration.

(c) Evaluate the integral from part (b).

Solution:

(b) From the figure we see that the region \( \mathcal{R} \) is bounded above by \( y = 4 \) and below by \( y = \sqrt{x} \) (obtained by solving \( x = y^2 \) for \( y \) in terms of \( x \)). The projection of \( \mathcal{R} \) onto the \( x \)-axis is the interval \( 0 \leq x \leq 16 \). Upon changing the order of integration we get the double integral:

\[
\int_0^{16} \int_{\sqrt{x}}^4 \frac{x^3}{4 - \sqrt{x}} \, dy \, dx
\]

(c) The integral from part (b) is evaluated as follows:

\[
\int_0^{16} \int_{\sqrt{x}}^4 \frac{x^3}{4 - \sqrt{x}} \, dy \, dx = \int_0^{16} \frac{x^3}{4 - \sqrt{x}} \left[ y \right]_{\sqrt{x}}^4 \, dx
\]

\[
= \int_0^{16} \frac{x^3}{4 - \sqrt{x}} (4 - \sqrt{x}) \, dx
\]

\[
= \int_0^{16} x^3 \, dx
\]

\[
= \left[ \frac{1}{4} x^4 \right]_0^{16}
\]

\[
= \frac{1}{4} (16)^4
\]

\[
= 16384
\]
4. For the vector field \( \vec{F} = \langle yx^2, y^2 \rangle \), find the value of \( \int_C \vec{F} \cdot d\vec{s} \) where \( C \) is the portion of the parabola \( y = x^2 \) from \((0, 0)\) to \((1, 1)\).

**Solution:** We evaluate the vector line integral using the formula:

\[
\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F} \cdot \vec{r}'(t) \, dt
\]

A parameterization of \( C \) is \( \vec{r}(t) = \langle t, t^2 \rangle \), \( 0 \leq t \leq 1 \). The derivative is \( \vec{r}'(t) = \langle 1, 2t \rangle \). Using the fact that \( x = t \) and \( y = t^2 \) from the parameterization, the vector field \( \vec{F} \) written in terms of \( t \) is:

\[
\vec{F} = \langle yx^2, y^2 \rangle = \langle (t^2)(t), (t^2)^2 \rangle = \langle t^4, t^4 \rangle
\]

Thus, the value of the line integral is:

\[
\int_C \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F} \cdot \vec{r}'(t) \, dt
\]

\[
= \int_0^1 \langle t^4, t^4 \rangle \cdot \langle 1, 2t \rangle \, dt
\]

\[
= \int_0^1 (t^4 + 2t^5) \, dt
\]

\[
= \left[ \frac{1}{5}t^5 + \frac{1}{3}t^6 \right]_0^1
\]

\[
= \left[ \frac{1}{5}(1)^5 + \frac{1}{3}(1)^6 \right] - \left[ \frac{1}{5}(0)^5 + \frac{1}{3}(0)^6 \right]
\]

\[
= \frac{8}{15}
\]
Math 210, Exam 2, Fall 2010  
Problem 5 Solution

5. Consider the vector field \( \mathbf{F} = \langle ax^2 y + 8xy^2 - 4, bx^2 y - 2x^3 - 1 \rangle \) where \( a \) and \( b \) are constants.

(a) Find the values of \( a \) and \( b \) for which \( \mathbf{F} \) is conservative.

(b) For the values of \( a \) and \( b \) from part (a), find a potential function \( \varphi(x, y) \) such that \( \mathbf{F} = \nabla \varphi \).

Solution:

(a) In order for the vector field \( \mathbf{F} = \langle f(x, y), g(x, y) \rangle \) to be conservative, it must be the case that:

\[
\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}
\]

Using \( f(x, y) = ax^2 y + 8xy^2 - 4 \) and \( g(x, y) = bx^2 y - 2x^3 - 1 \) we get:

\[
\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \\
ax^2 + 16xy = 2bxy - 6x^2 \\
ax^2 + 6x^2 = 2bxy - 16xy \\
(a + 6)x^2 = (2b - 16)xy
\]

In order for the above equation to be satisfied for all pairs \((x, y)\), it must be the case that \( a + 6 = 0 \) and \( 2b - 16 = 0 \) which give us \[ a = -6 \] and \[ b = 8 \].

(b) If \( \mathbf{F} = \nabla \varphi \), then it must be the case that:

\[
\frac{\partial \varphi}{\partial x} = f(x, y) \tag{1} \\
\frac{\partial \varphi}{\partial y} = g(x, y) \tag{2}
\]

Using \( f(x, y) = -6x^2 y + 8xy^2 - 4 \) and integrating both sides of Equation (1) with respect to \( x \) we get:

\[
\frac{\partial \varphi}{\partial x} = f(x, y) \\
\frac{\partial \varphi}{\partial x} = -6x^2 y + 8xy^2 - 4 \\
\int \frac{\partial \varphi}{\partial x} \, dx = \int (-6x^2 y + 8xy^2 - 4) \, dx \\
\varphi(x, y) = -2x^3 y + 4x^2 y^2 - 4x + h(y)
\]

\[ \text{(3)} \]
We obtain the function $h(y)$ using Equation (2). Using $g(x, y) = 8x^2y - 2x^3 - 1$ we get the equation:

$$\frac{\partial \varphi}{\partial y} = g(x, y)$$

$$\frac{\partial \varphi}{\partial y} = 8x^2y - 2x^3 - 1$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\frac{\partial}{\partial y} (-2x^3y + 4x^2y^2 - 4x + h(y)) = 8x^2y - 2x^3 - 1$$

$$-2x^3 + 8x^2y + h'(y) = 8x^2y - 2x^3 - 1$$

$$h'(y) = -1$$

Now integrate both sides with respect to $y$ to get:

$$\int h'(y) \, dy = \int -1 \, dy$$

$$h(y) = -y + C$$

Letting $C = 0$, we find that a potential function for $\mathbf{F}$ is:

$$\varphi(x, y) = -2x^3y + 4x^2y^2 - 4x - y$$
6. Compute the surface area of the part of the paraboloid \( z = 4 - x^2 - y^2 \) that lies in the region \( \{ (x, y, z) \mid z \geq 3, \ x \geq 0 \} \).

**Solution:** The formula for surface area we will use is:

\[
S = \iint_{S} dS = \iint_{R} \left| \mathbf{t}_u \times \mathbf{t}_v \right| \, dA
\]

where the function \( \mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \) with domain \( \mathcal{R} \) is a parameterization of the surface \( S \) and the vectors \( \mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} \) and \( \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} \) are the tangent vectors.

We begin by finding a parameterization of the paraboloid. Let \( x = u \cos(v) \) and \( y = u \sin(v) \), where we define \( u \) to be nonnegative. Then,

\[
\begin{align*}
z &= 4 - x^2 - y^2 \\
&= 4 - (u \cos(v))^2 - (u \sin(v))^2 \\
&= 4 - u^2 \cos^2(v) - u^2 \sin^2(v) \\
&= 4 - u^2
\end{align*}
\]

Thus, we have \( \mathbf{r}(u, v) = \langle u \cos(v), u \sin(v), 4 - u^2 \rangle \). To find the domain \( \mathcal{R} \), we must interpret the inequalities \( z \geq 0 \) and \( x \geq 0 \) in terms of the new variables \( u \) and \( v \). From the first inequality we find that:

\[
\begin{align*}
z &\geq 0 \\
4 - u^2 &\geq 0 \\
u^2 &\leq 4 \\
0 &\leq u \leq 2
\end{align*}
\]

noting that, by definition, \( u \) must be nonnegative. From the second inequality we find that:

\[
\begin{align*}
x &\geq 0 \\
u \cos(v) &\geq 0 \\
\cos(v) &\geq 0 \\
-\frac{\pi}{2} &\leq v \leq \frac{\pi}{2}
\end{align*}
\]

noting that \( \cos(v) \geq 0 \) implies that \( v \) is an angle in either Quadrant I or IV. Therefore, a parameterization of \( S \) is:

\[
\begin{align*}
\mathbf{r}(u, v) &= \langle u \cos(v), u \sin(v), 4 - u^2 \rangle, \\
\mathcal{R} &= \left\{ (u, v) \mid 0 \leq u \leq 2, \ -\frac{\pi}{2} \leq v \leq \frac{\pi}{2} \right\}
\end{align*}
\]
The tangent vectors $\vec{t}_u$ and $\vec{t}_v$ are then:

$$\vec{t}_u = \frac{\partial \vec{r}}{\partial u} = \langle \cos(v), \sin(v), -2u \rangle$$

$$\vec{t}_v = \frac{\partial \vec{r}}{\partial v} = \langle -u \sin(v), u \cos(v), 0 \rangle$$

The cross product of these vectors is:

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(v) & \sin(v) & -2u \\ -u \sin(v) & u \cos(v) & 0 \end{vmatrix} = 2u^2 \cos(v) \hat{i} + 2u^2 \sin(v) \hat{j} + u \hat{k}$$

The magnitude of the cross product is:

$$\left| \vec{t}_u \times \vec{t}_v \right| = \sqrt{(2u^2 \cos(v))^2 + (2u^2 \sin(v))^2 + u^2}$$

$$= \sqrt{4u^4 \cos^2(v) + 4u^4 \sin^2(v) + u^2}$$

$$= \sqrt{4u^4 + u^2}$$

$$= u \sqrt{4u^2 + 1}$$

We can now compute the surface area.

$$S = \iint_{\mathcal{R}} \left| \vec{t}_u \times \vec{t}_v \right| \, dA$$

$$= \int_0^2 \int_{-\pi/2}^{\pi/2} u \sqrt{4u^2 + 1} \, dv \, du$$

$$= \int_0^2 u \sqrt{4u^2 + 1} \left[ v \right]_{-\pi/2}^{\pi/2} \, du$$

$$= \int_0^2 u \sqrt{4u^2 + 1} \left[ \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right] \, du$$

$$= \int_0^2 \pi u \sqrt{4u^2 + 1} \, du$$

$$= \left[ \frac{\pi}{12} (4u^2 + 1)^{3/2} \right]_0^2$$

$$= \left[ \frac{\pi}{12} (4(2)^2 + 1)^{3/2} \right] - \left[ \frac{\pi}{12} (4(0)^2 + 1)^{3/2} \right]$$

$$= \frac{\pi}{12} (17)^{3/2} - \frac{\pi}{12} (1)^{3/2}$$

$$= \frac{\pi}{12} (17\sqrt{17} - 1)$$