1. Let \( f(x, y) = 3x^2 + xy + 2y^2 \). Find the partial derivatives \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \), at (1, 1), and find the best linear approximation of \( f \) at (1, 1) and use it to estimate \( f(1.1, 1.2) \).

**Solution:** The linearization of \( f(x, y) = 3x^2 + xy + 2y^2 \) about (1, 1) has the form:

\[
L(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)
\]

The first partial derivatives of \( f(x, y) \) are:

\[
\begin{align*}
f_x &= 6x + y \\
f_y &= x + 4y
\end{align*}
\]

At the point (1, 1) we have:

\[
\begin{align*}
f(1, 1) &= 3(1)^2 + (1)(1) + 2(1)^2 = 6 \\
f_x(1, 1) &= 6(1) + 1 = 7 \\
f_y(1, 1) &= 1 + 4(1) = 5
\end{align*}
\]

Thus, the linearization is:

\[
L(x, y) = 6 + 7(x - 1) + 5(y - 1)
\]

The value of \( f(1.1, 1.2) \) is estimated to be the value of \( L(1.1, 1.2) \):

\[
\begin{align*}
f(1.1, 1.2) &\approx L(1.1, 1.2) \\
f(1.1, 1.2) &\approx 6 + 7(1.1 - 1) + 5(1.2 - 1) \\
f(1.1, 1.2) &\approx 7.7
\end{align*}
\]
2. Find and classify the critical points of the function

\[ f(x, y) = x^3 - 3xy + y^3. \]

**Solution:** By definition, an interior point \((a, b)\) in the domain of \(f\) is a **critical point** of \(f\) if either

1. \(f_x(a, b) = f_y(a, b) = 0\), or
2. one (or both) of \(f_x\) or \(f_y\) does not exist at \((a, b)\).

The partial derivatives of \(f(x, y) = x^3 - 3xy + y^3\) are \(f_x = 3x^2 - 3y\) and \(f_y = -3x + 3y^2\). These derivatives exist for all \((x, y)\) in \(\mathbb{R}^2\). Thus, the critical points of \(f\) are the solutions to the system of equations:

\[
\begin{align*}
  f_x &= 3x^2 - 3y = 0, \\
  f_y &= -3x + 3y^2 = 0.
\end{align*}
\]

Solving Equation (1) for \(y\) we get:

\[ y = x^2 \] (3)

Substituting this into Equation (2) and solving for \(x\) we get:

\[
\begin{align*}
  -3x + 3y^2 &= 0 \\
  -3x + 3(x^2)^2 &= 0 \\
  -3x + 3x^4 &= 0 \\
  3x(x^3 - 1) &= 0
\end{align*}
\]

We observe that the above equation is satisfied if either \(x = 0\) or \(x^3 - 1 = 0 \iff x = 1\). We find the corresponding \(y\)-values using Equation (3): \(y = x^2\).

- If \(x = 0\), then \(y = 0^2 = 0\).
- If \(x = 1\), then \(y = 1^2 = 1\).

Thus, the critical points are \((0, 0)\) and \((1, 1)\).

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of \(f\) are:

\[ f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3 \]
The discriminant function $D(x, y)$ is then:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2$$
$$D(x, y) = (6x)(6y) - (-3)^2$$
$$D(x, y) = 36xy - 9$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

<table>
<thead>
<tr>
<th>$(a, b)$</th>
<th>$D(a, b)$</th>
<th>$f_{xx}(a, b)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>$-9$</td>
<td>$0$</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>$27$</td>
<td>$6$</td>
<td>Local Minimum</td>
</tr>
</tbody>
</table>

Recall that $(a, b)$ is a saddle point if $D(a, b) < 0$ and that $(a, b)$ corresponds to a local minimum of $f$ if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$.

Figure 1: Picture above are level curves of $f(x, y)$. Darker colors correspond to smaller values of $f(x, y)$. It is apparent that $(0, 0)$ is a saddle point and $(1, 1)$ corresponds to a local minimum.
3. Sketch the region of integration for the integral $\int_0^4 \int_{\sqrt{y}}^2 \sin \left( x^3 \right) \, dx \, dy$. Compute the integral.

**Solution:** The region of integration $\mathcal{R}$ is sketched below:

First, we recognize that $\sin \left( x^3 \right)$ has no simple antiderivative. Therefore, we must change the order of integration to evaluate the integral. The region $\mathcal{R}$ can be described as follows:

$$\mathcal{R} = \{(x, y) : 0 \leq y \leq x^2, \ 0 \leq x \leq 2\}$$

where $y = 0$ is the bottom curve and $y = x^2$ is the top curve, obtained by solving the equation $x = \sqrt{y}$ for $y$ in terms of $x$. Therefore, the value of the integral is:

$$\int_0^4 \int_{\sqrt{y}}^2 \sin \left( x^3 \right) \, dx \, dy = \int_0^2 \int_0^{x^2} \sin \left( x^3 \right) \, dy \, dx$$

$$= \int_0^2 \sin \left( x^3 \right) \left[ y \right]_0^{x^2} \, dx$$

$$= \int_0^2 x^2 \sin \left( x^3 \right) \, dx$$

$$= \left[ -\frac{1}{3} \cos \left( x^3 \right) \right]_0^2$$

$$= \left[ -\frac{1}{3} \cos \left( 2^3 \right) \right] - \left[ -\frac{1}{3} \cos \left( 0^3 \right) \right]$$

$$= -\frac{1}{3} \cos(8) + \frac{1}{3}$$
4. Find the minimum and maximum of the function \( f(x, y, z) = x + y - z \) on the ellipsoid

\[
R = \left\{ (x, y, z) \mid \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \right\}
\]

**Solution:** We find the minimum and maximum using the method of Lagrange Multipliers. First, we recognize that \( R \) is compact which guarantees the existence of absolute extrema of \( f \). Then, let \( g(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \). We look for solutions to the following system of equations:

\[ f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x, y, z) = 1 \]

which, when applied to our functions \( f \) and \( g \), give us:

\[
\begin{align*}
1 &= \lambda \left( \frac{2x}{4} \right) \\
1 &= \lambda \left( \frac{2y}{9} \right) \\
-1 &= \lambda \left( 2z \right) \\
\frac{x^2}{4} + \frac{y^2}{9} + z^2 &= 1
\end{align*}
\]

To solve the system of equations, we first solve Equations (1)-(3) for the variables \( x, y, \) and \( z \) in terms of \( \lambda \) to get:

\[
\begin{align*}
x &= \frac{4}{2\lambda}, \quad y = \frac{9}{2\lambda}, \quad z = -\frac{1}{2\lambda}
\end{align*}
\]

We then plug Equations (5) into Equation (4) and simplify.

\[
\frac{\left( \frac{4}{2\lambda} \right)^2}{4} + \frac{\left( \frac{9}{2\lambda} \right)^2}{9} + \left( -\frac{1}{2\lambda} \right)^2 = 1
\]

\[
\frac{16}{4} + \frac{81}{9} + \frac{1}{4\lambda^2} = 1
\]
At this point we multiply both sides of the equation by $4\lambda^2$ to get:

$$4\lambda^2 \left( \frac{16}{4 \lambda^2} + \frac{81}{9 \lambda^2} + \frac{1}{4 \lambda^2} \right) = 4\lambda^2 (1)$$

$$\frac{16}{4} + \frac{81}{9} + 1 = 4\lambda^2$$

$$4 + 9 + 1 = 4\lambda^2$$

$$\lambda^2 = \frac{7}{2}$$

$$\lambda = \pm \sqrt{\frac{7}{2}}$$

$$\lambda = \pm \sqrt{\frac{14}{2}}$$

- When $\lambda = \sqrt{\frac{14}{2}}$, Equations (5) give us the first candidate for the location of an extreme value:
  $$x = \frac{4\sqrt{14}}{14}, \quad y = \frac{9\sqrt{14}}{14}, \quad z = -\frac{\sqrt{14}}{14}$$

- When $\lambda = -\sqrt{\frac{14}{2}}$, Equations (5) give us the first candidate for the location of an extreme value:
  $$x = -\frac{4\sqrt{14}}{14}, \quad y = -\frac{9\sqrt{14}}{14}, \quad z = \frac{\sqrt{14}}{14}$$

Evaluating $f(x, y, z)$ at these points we find that:

$$f \left( \frac{4\sqrt{14}}{14}, \frac{9\sqrt{14}}{14}, -\frac{\sqrt{14}}{14} \right) = \sqrt{14}$$

$$f \left( -\frac{4\sqrt{14}}{14}, -\frac{9\sqrt{14}}{14}, \frac{\sqrt{14}}{14} \right) = -\sqrt{14}$$

Therefore, the absolute maximum value of $f$ on $R$ is $\sqrt{14}$ and the absolute minimum of $f$ on $R$ is $-\sqrt{14}$.

**Note:** The level surfaces $f(x, y, z) = \sqrt{14}$ and $f(x, y, z) = -\sqrt{14}$ are planes tangent to the ellipsoid at the critical points.
Math 210, Exam 2, Practice Fall 2009
Problem 5 Solution

5. Find the tangent plane to the surface:

\[ S = \{ (x, y, z) : x^2 + y^3 - 2z = 1 \} \]

at the point (1, 2, 4).

Solution: Let \( F(x, y, z) = x^2 + y^3 - 2z = 1 \) be the equation for the surface. We use the following formula for the equation for the tangent plane:

\[ F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0 \]

because the equation for the surface is given in implicit form. Note that \( \mathbf{n} = \nabla F(a, b, c) = \langle F_x(a, b, c), F_y(a, b, c), F_z(a, b, c) \rangle \) is a vector normal to the surface \( F(x, y, z) = C \) and, thus, to the tangent plane at the point \((a, b, c)\) on the surface.

The partial derivatives of \( F(x, y, z) = x^2 + y^3 - 2z \) are:

\[ F_x = 2x, \quad F_y = 3y^2, \quad F_z = -2 \]

Evaluating these derivatives at \((1, 2, 4)\) we get:

\[ F_x(1, 2, 4) = 2(1) = 2 \]
\[ F_y(1, 2, 4) = 3(2)^2 = 12 \]
\[ F_z(1, 2, 4) = -2 \]

Thus, the tangent plane equation is:

\[ 2(x - 1) + 12(y - 2) - 2(z - 4) = 0 \]
6. Let \( F(x, y, z) = 3x^2 + y^2 - 4z^2 \). Find the equation of the tangent plane to the level surface \( F(x, y, z) = 1 \) at the point \((1, -4, 3)\).

**Solution**: We use the following formula for the equation for the tangent plane:

\[
F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0
\]

because the equation for the surface is given in **implicit** form. Note that \( \vec{n} = \nabla F(a, b, c) = \langle F_x(a, b, c), F_y(a, b, c), F_z(a, b, c) \rangle \) is a vector normal to the surface \( F(x, y, z) = C \) and, thus, to the tangent plane at the point \((a, b, c)\) on the surface.

The partial derivatives of \( F(x, y, z) = 3x^2 + y^2 - 4z^2 \) are:

\[
F_x = 6x, \quad F_y = 2y, \quad F_z = -8z
\]

Evaluating these derivatives at \((1, -4, 3)\) we get:

\[
F_x(1, -4, 3) = 6(1) = 6 \\
F_y(1, -4, 3) = 2(-4) = -8 \\
F_z(1, -4, 3) = -8(3) = -24
\]

Thus, the tangent plane equation is:

\[
6(x - 1) - 8(y + 4) - 24(z - 3) = 0
\]
Math 210, Exam 2, Practice Fall 2009
Problem 7 Solution

7. Let \( f(x, y) = \frac{1}{3}x^3 + y^2 - xy \). Find all critical points of \( f(x, y) \) and classify each as a local maximum, local minimum, or saddle point.

**Solution:** By definition, an interior point \((a, b)\) in the domain of \( f \) is a **critical point** of \( f \) if either

1. \( f_x(a, b) = f_y(a, b) = 0 \), or
2. one (or both) of \( f_x \) or \( f_y \) does not exist at \((a, b)\).

The partial derivatives of \( f(x, y) = \frac{1}{3}x^3 + y^2 - xy \) are \( f_x = x^2 - y \) and \( f_y = 2y - x \). These derivatives exist for all \((x, y)\) in \( \mathbb{R}^2 \). Thus, the critical points of \( f \) are the solutions to the system of equations:

\[
\begin{align*}
  f_x &= x^2 - y = 0 \quad (1) \\
  f_y &= 2y - x = 0 \quad (2)
\end{align*}
\]

Solving Equation (1) for \( y \) we get:

\[
y = x^2 \quad (3)
\]

Substituting this into Equation (2) and solving for \( x \) we get:

\[
\begin{align*}
  2y - x &= 0 \\
  2x^2 - x &= 0 \\
  x(2x - 1) &= 0 \\
  \iff x &= 0 \text{ or } x = \frac{1}{2}
\end{align*}
\]

We find the corresponding \( y \)-values using Equation (3): \( y = x^2 \).

- If \( x = 0 \), then \( y = 0^2 = 0 \).
- If \( x = \frac{1}{2} \), then \( y = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \).

Thus, the critical points are \((0, 0)\) and \((\frac{1}{2}, \frac{1}{4})\).

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of \( f \) are:

\[
\begin{align*}
  f_{xx} &= 2x, \quad f_{yy} = 2, \quad f_{xy} = -1
\end{align*}
\]

The discriminant function \( D(x, y) \) is then:

\[
D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 - (-1)^2 = 4x - 1
\]

The values of \( D(x, y) \) at the critical points and the conclusions of the Second Derivative Test are shown in the table below.
Recall that \((a, b)\) is a saddle point if \(D(a, b) < 0\) and that \((a, b)\) corresponds to a local minimum of \(f\) if \(D(a, b) > 0\) and \(f_{xx}(a, b) > 0\).

<table>
<thead>
<tr>
<th>((a, b))</th>
<th>(D(a, b))</th>
<th>(f_{xx}(a, b))</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>(-1)</td>
<td>0</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>((\frac{1}{2}, \frac{1}{4}))</td>
<td>1</td>
<td>1</td>
<td>Local Minimum</td>
</tr>
</tbody>
</table>

Figure 1: Picture above are level curves of \(f(x, y)\). Darker colors correspond to smaller values of \(f(x, y)\). It is apparent that \((0, 0)\) is a saddle point and \((\frac{1}{2}, \frac{1}{4})\) corresponds to a local minimum.
8. Find the minimum and maximum of the function $f(x, y) = x^2 - y$ subject to the condition $x^2 + y^2 = 4$.

**Solution:** We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $x^2 + y^2 = 4$ is compact which guarantees the existence of absolute extrema of $f$. Then, let $g(x, y) = x^2 + y^2 = 4$. We look for solutions to the following system of equations:

$$
\begin{align*}
    f_x &= \lambda g_x, \\
    f_y &= \lambda g_y, \\
    g(x, y) &= 4
\end{align*}
$$

which, when applied to our functions $f$ and $g$, give us:

$$
\begin{align*}
    2x &= \lambda (2x) \\
    -1 &= \lambda (2y) \\
    x^2 + y^2 &= 4
\end{align*}
$$

We begin by noting that Equation (1) gives us:

$$
\begin{align*}
    2x &= \lambda (2x) \\
    2x - \lambda (2x) &= 0 \\
    2x(1 - \lambda) &= 0
\end{align*}
$$

From this equation we either have $x = 0$ or $\lambda = 1$. Let’s consider each case separately.

**Case 1:** Let $x = 0$. We find the corresponding $y$-values using Equation (3).

$$
\begin{align*}
    x^2 + y^2 &= 4 \\
    0^2 + y^2 &= 4 \\
    y^2 &= 4 \\
    y &= \pm 2
\end{align*}
$$

Thus, the points of interest are $(0, 2)$ and $(0, -2)$.

**Case 2:** Let $\lambda = 1$. Plugging this into Equation (2) we get:

$$
\begin{align*}
    -1 &= \lambda (2y) \\
    -1 &= 1(2y) \\
    y &= -\frac{1}{2}
\end{align*}
$$

We find the corresponding $x$-values using Equation (3).

$$
\begin{align*}
    x^2 + y^2 &= 4 \\
    x^2 + \left(-\frac{1}{2}\right)^2 &= 4 \\
    x^2 + \frac{1}{4} &= 4 \\
    x^2 &= \frac{15}{4} \\
    x &= \pm \frac{\sqrt{15}}{2}
\end{align*}
$$

Thus, the points of interest are $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$. 

We now evaluate \( f(x, y) = x^2 - y \) at each point of interest obtained by Cases 1 and 2.

\[
\begin{align*}
f(0, 2) &= -2 \\
f(0, -2) &= 2 \\
f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) &= \frac{17}{4} \\
f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) &= \frac{17}{4}
\end{align*}
\]

From the values above we observe that \( f \) attains an absolute maximum of \( \frac{17}{4} \) and an absolute minimum of \( -2 \).

Figure 1: Shown in the figure are the level curves of \( f(x, y) = x^2 - y \) and the circle \( x^2 + y^2 = 4 \) (thick, black curve). Darker colors correspond to smaller values of \( f(x, y) \). Notice that (1) the parabola \( f(x, y) = x^2 - y = \frac{17}{4} \) is tangent to the circle at the points \( \left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) \) and \( \left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) \) which correspond to the absolute maximum and (2) the parabola \( f(x, y) = x^2 - y = -2 \) is tangent to the circle at the point \((0, 2)\) which corresponds to the absolute minimum.
9. Use polar coordinates to find the volume of the region bounded by the paraboloid \( z = 1 - x^2 - y^2 \) in the first octant \( x \geq 0, y \geq 0, z \geq 0 \).

**Solution:** The volume formula we use is:

\[
V = \iiint_D (1 - x^2 - y^2) \, dA
\]

where \( D \) is the projection of the paraboloid onto the first quadrant in the \( xy \)-plane. We are asked to use polar coordinates:

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad dA = r \, dr \, d\theta
\]

1. First, we describe the region \( D \). Since \( z \geq 0 \) and \( z = 1 - x^2 - y^2 \) we know that:

\[
1 - x^2 - y^2 \geq 0 \quad \text{and} \quad x^2 + y^2 \leq 1
\]

Since the projection is in the first quadrant, the region \( D \) can be described in rectangular coordinates as:

\[
D = \{(x, y) : x^2 + y^2 \leq 1, \ x \geq 0, \ y \geq 0\}
\]

or, equivalently, in polar coordinates as:

\[
D = \{(r, \theta) : 0 \leq r \leq 1, \ 0 \leq \theta \leq \frac{\pi}{2}\}
\]

2. Then, using the polar coordinate equations, the paraboloid \( z = 1 - x^2 - y^2 \) can be written in polar coordinates as:

\[
z = 1 - r^2
\]

3. Finally, we compute the volume as follows:

\[
V = \iiint_D (1 - x^2 - y^2) \, dA\\
= \int_0^{\pi/2} \int_0^1 \left(1 - r^2\right) r \, dr \, d\theta\\
= \int_0^{\pi/2} \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 \, d\theta\\
= \int_0^{\pi/2} \frac{1}{4} \, d\theta\\
= \left[\frac{1}{4}\theta\right]_0^{\pi/2}\\
= \frac{\pi}{8}
\]
10. Find the minimum and maximum of the function 
\[ f(x, y, z) = x^2 - y^2 + 2z^2 \]
on the surface of the sphere defined by the equation \( x^2 + y^2 + z^2 = 1 \).

**Solution:** We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that the sphere is compact and that \( f(x, y, z) \) is continuous on the sphere, which guarantees the existence of absolute extrema of \( f \). Then, let \( g(x, y, z) = x^2 + y^2 + z^2 = 1 \). We look for solutions to the following system of equations:

\[
\begin{align*}
    f_x &= \lambda g_x, & f_y &= \lambda g_y, & f_z &= \lambda g_z, & g(x, y, z) &= 1
\end{align*}
\]

which, when applied to our functions \( f \) and \( g \), give us:

\[
\begin{align*}
    2x &= \lambda (2x) \quad (1) \\
    -2y &= \lambda (2y) \quad (2) \\
    4z &= \lambda (2z) \quad (3) \\
    x^2 + y^2 + z^2 &= 1 \quad (4)
\end{align*}
\]

From Equation (1) we can have either \( x = 0 \) or \( \lambda = 1 \).

- If \( x = 0 \) then we turn to Equation (2). In this case we either have \( y = 0 \) or \( \lambda = -1 \).
  - Suppose \( y = 0 \). Plugging \( x = 0 \) and \( y = 0 \) into Equation (4) we get:
    \[
    \begin{align*}
        x^2 + y^2 + z^2 &= 1 \\
        0^2 + 0^2 + z^2 &= 1 \\
        z^2 &= 1 \\
        z &= \pm 1
    \end{align*}
    \]
    Thus, the points of interest are \((0, 0, 1)\) and \((0, 0, -1)\).
  - Now suppose \( \lambda = -1 \). Then Equation (3) gives us:
    \[
    \begin{align*}
        4z &= \lambda (2z) \\
        4z &= (-1)(2z) \\
        6z &= 0 \\
        z &= 0
    \end{align*}
    \]
    Plugging \( x = 0 \) and \( z = 0 \) into Equation (4) we get:
    \[
    \begin{align*}
        x^2 + y^2 + z^2 &= 1 \\
        0^2 + y^2 + 0^2 &= 1 \\
        y^2 &= 1 \\
        y &= \pm 1
    \end{align*}
    \]
    Thus, the points of interest are \((0, 1, 0)\) and \((0, -1, 0)\).

1
• If $\lambda = 1$ then Equations (2) and (3) give us:

\[
\begin{align*}
-2y &= \lambda(2y) \\
-2y &= (1)(2y) \\
-4y &= 0 \\
y &= 0
\end{align*}
\]

\[
\begin{align*}
4z &= \lambda(2z) \\
4z &= (1)(2z) \\
2z &= 0 \\
z &= 0
\end{align*}
\]

Plugging $y = 0$ and $z = 0$ into Equation (4) we get:

\[
\begin{align*}
x^2 + y^2 + z^2 &= 1 \\
x^2 + 0^2 + 0^2 &= 1 \\
x^2 &= 1 \\
x &= \pm 1
\end{align*}
\]

Thus, the points of interest are $(1, 0, 0)$ and $(-1, 0, 0)$.

Evaluating $f(x, y, z)$ at all points of interest we find that:

\[
\begin{align*}
f(1, 0, 0) &= 1 \\
f(-1, 0, 0) &= 1 \\
f(0, 1, 0) &= -1 \\
f(0, -1, 0) &= -1 \\
f(0, 0, 1) &= 2 \\
f(0, 0, -1) &= 2
\end{align*}
\]

Therefore, the absolute maximum value of $f$ is 2 and the absolute minimum of $f$ is $-1$. 

11. Using cylindrical coordinates, compute
\[ \iiint_W \left( x^2 + y^2 \right)^{1/2} \, dV \]
where \( W \) is the region within the cylinder \( x^2 + y^2 \leq 4 \) and \( 0 \leq z \leq y \).

**Solution:** The region \( W \) is plotted below.

In cylindrical coordinates, the equations for the cylinder \( x^2 + y^2 = 4 \) and the plane \( z = y \) are:

- **Cylinder:** \( r = 2 \)
- **Plane:** \( z = r \sin \theta \)

Furthermore, we can write the integrand in cylindrical coordinates as:

\[ f(x, y, z) = \left( x^2 + y^2 \right)^{1/2} \]
\[ f(r, \theta, z) = r \]

The projection of \( W \) onto the \( xy \)-plane is the half-disk \( 0 \leq r \leq 2, \ 0 \leq \theta \leq \pi \). Using the fact that \( dV = r \, dz \, dr \, d\theta \) in cylindrical coordinates, the value of the integral is:
\[ \iiint_{W} (x^2 + y^2)^{1/2} \, dV = \int_{0}^{\pi} \int_{0}^{2} \int_{0}^{r \sin \theta} r^2 \, dz \, dr \, d\theta \]
\[ = \int_{0}^{\pi} \int_{0}^{2} r^3 \sin \theta \, dr \, d\theta \]
\[ = \int_{0}^{\pi} \sin \theta \left[ \frac{1}{4} r^4 \right]_{0}^{2} \, d\theta \]
\[ = 4 \int_{0}^{\pi} \sin \theta \, d\theta \]
\[ = 4 \left[ -\cos \theta \right]_{0}^{\pi} \]
\[ = 4 \left[ -\cos \pi + \cos 0 \right] \]
\[ = 8 \]
12. Compute the integral \( \iiint_B x^2 \, dV \), where \( B \) is the unit ball

\[ B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\} \]

**Solution:** Due to the fact that \( B \) is a ball of radius 1, we use Spherical Coordinates to evaluate the integral. In Spherical Coordinates, the equation for the sphere is \( \rho = 1 \) and the integrand is:

\[ f(x, y, z) = x^2 \]
\[ f(\rho, \phi, \theta) = (\rho \sin \phi \cos \theta)^2 \]

Using the fact that \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \) in Spherical Coordinates, the value of the integral is:

\[
\iiint_B x^2 \, dV = \int_0^{2\pi} \int_0^\pi \int_0^1 (\rho \sin \phi \cos \theta)^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
= \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^4 \sin^3 \phi \cos^2 \theta \, d\rho \, d\phi \, d\theta \\
= \int_0^{2\pi} \int_0^\pi \sin^3 \phi \cos^2 \theta \left[ \frac{1}{5} \rho^5 \right]_0^1 \, d\phi \, d\theta \\
= \frac{1}{5} \int_0^{2\pi} \int_0^\pi \sin^3 \phi \cos^2 \theta \, d\phi \, d\theta \\
= \frac{1}{5} \int_0^{2\pi} \cos^2 \theta \left[ \frac{1}{3} \cos^3 \phi - \cos \phi \right]_0^\pi \, d\theta \\
= \frac{1}{5} \int_0^{2\pi} \cos^2 \theta \left[ \left( \frac{1}{3} \cos^3 \pi - \cos \pi \right) - \left( \frac{1}{3} \cos^3 0 - \cos 0 \right) \right] \, d\theta \\
= \frac{1}{5} \int_0^{2\pi} \frac{4}{3} \cos^2 \theta \, d\theta \\
= \frac{4}{15} \left[ \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \\
= \frac{4}{15} \left[ \left( \frac{1}{2}(2\pi) + \frac{1}{4}\sin(4\pi) \right) - \left( \frac{1}{2}(0) + \frac{1}{4}\sin(0) \right) \right] \\
= \frac{4\pi}{15}
\]
13. Find the volume of the region bounded below and above by the surfaces \( z = x^2 + y^2 \) and \( z = 2 - x^2 - y^2 \).

**Solution:** The region is plotted below.

The volume may be computed using either a double integral or a triple integral. Using a triple integral, the formula is:

\[
V = \iiint_R 1 \, dV
\]

Due to the shape of the boundary, we will use Cylindrical Coordinates. The paraboloids can be written in Cylindrical Coordinates as:

- Paraboloid 1: \( z = r^2 \)
- Paraboloid 2: \( z = 2 - r^2 \)

The region \( R \) is bounded above by \( z = 2 - r^2 \) and below by \( z = r^2 \). The projection of \( R \) onto the \( xy \)-plane is the disk \( 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi \). The radius of the disk is obtained by determining the intersection of the two surfaces:

- \( z = z \)
- \( r^2 = 2 - r^2 \)
- \( r^2 = 1 \)
- \( r = 1 \)
Using the fact that $dV = r \, dz \, dr \, d\theta$ in Cylindrical Coordinates, the volume is:

\[
V = \iiint_R 1 \, dV \\
= \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r \, dz \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^1 r \left[ z \right]_{r^2}^{2-r^2} \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^1 r \left( 2 - r^2 - r^2 \right) \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^1 (2r - 2r^3) \, dr \, d\theta \\
= \int_0^{2\pi} \left[ r^2 - \frac{1}{2} r^4 \right]_0^1 \, d\theta \\
= \frac{1}{2} \int_0^{2\pi} \, d\theta \\
= \frac{1}{2} \left[ \theta \right]_0^{2\pi} \\
= \left[ \frac{\pi}{2} \right]
\]
Math 210, Exam 2, Practice Fall 2009
Problem 14 Solution

14. Let \( f(x, y) = e^{xy} \) and \((r, \theta)\) be polar coordinates. Find \( \frac{\partial f}{\partial r} \). Express your answer in terms of the variables \( x \) and \( y \).

Solution: First, the equations for \( x \) and \( y \) in polar coordinates are defined as:

\[
x = r \cos \theta, \quad y = r \sin \theta
\]  

(1)

Using the Chain Rule, the derivative \( \frac{\partial f}{\partial r} \) can be expressed as follows:

\[
\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}
\]  

(2)

The partial derivatives on the right hand side of the above equation are:

\[
\frac{\partial f}{\partial x} = ye^{xy}, \quad \frac{\partial x}{\partial r} = \cos \theta
\]

\[
\frac{\partial f}{\partial y} = xe^{xy}, \quad \frac{\partial y}{\partial r} = \sin \theta
\]

Plugging these into Equation (2) and using Equations (1) we get:

\[
\frac{\partial f}{\partial r} = ye^{xy} \cos \theta + xe^{xy} \sin \theta
\]

\[
\frac{\partial f}{\partial r} = e^{xy}(y \cos \theta + x \sin \theta)
\]

Using the fact that:

\[
\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}, \quad r = \sqrt{x^2 + y^2}
\]

we can write our answer in terms of \( x \) and \( y \):

\[
\frac{\partial f}{\partial r} = e^{xy}(y \cos \theta + x \sin \theta)
\]

\[
\frac{\partial f}{\partial r} = ye^{xy} \left( y \cdot \frac{x}{r} + x \cdot \frac{y}{r} \right)
\]

\[
\frac{\partial f}{\partial r} = ye^{xy} \left( y \cdot \frac{x}{\sqrt{x^2 + y^2}} + x \cdot \frac{y}{\sqrt{x^2 + y^2}} \right)
\]

\[
\frac{\partial f}{\partial r} = \frac{2xy}{\sqrt{x^2 + y^2}} e^{xy}
\]
15. Compute the average value of the function \( f(x, y) = 2 + x - y \) on the quarter disk \( A = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\} \).

**Solution:** We use the following formula to compute the average value of \( f \):

\[
\bar{f} = \frac{\iint_A f(x, y) \, dA}{\iint_A 1 \, dA}
\]

Since the region \( A \) is a quarter circle, we use polar coordinates: \( x = r \cos \theta, y = r \sin \theta, \quad dA = r \, dr \, d\theta \). The region \( A \) can then be described as:

\[
A = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}
\]

and the function \( f \) written in polar coordinates is:

\[
f(r, \theta) = 2 + r \cos \theta - r \sin \theta
\]

The double integral of \( f \) over \( A \) is then:

\[
\iint_A f(x, y) \, dA = \int_0^{\pi/2} \int_0^1 (2 + r \cos \theta - r \sin \theta) \, r \, dr \, d\theta
\]

\[
= \int_0^{\pi/2} \left[ r^2 + \frac{1}{3} r^3 \cos \theta - \frac{1}{3} r^3 \sin \theta \right]_0^1 \, d\theta
\]

\[
= \int_0^{\pi/2} \left( 1 + \frac{1}{3} \cos \theta - \frac{1}{3} \sin \theta \right) \, d\theta
\]

\[
= \left[ \theta + \frac{1}{3} \sin \theta + \frac{1}{3} \cos \theta \right]_0^{\pi/2}
\]

\[
= \left[ \frac{\pi}{2} + \frac{1}{3} \sin \frac{\pi}{2} + \frac{1}{3} \cos \frac{\pi}{2} \right] - \left[ 0 + \frac{1}{3} \sin 0 + \frac{1}{3} \cos 0 \right]
\]

\[
= \frac{\pi}{2}
\]

We recognize that the double integral \( \iint_A 1 \, dA \) represents the area of \( A \). Since \( A \) is a quarter circle of radius 1, the area is \( \frac{\pi}{4} \). Thus, the average value of \( f \) is:

\[
\bar{f} = \frac{\iint_A f(x, y) \, dA}{\iint_A 1 \, dA} = \frac{\frac{\pi}{2}}{\frac{\pi}{4}} = 2
\]
16. Compute the integral
\[ \int\int_{D} \frac{x}{y+1} \, dA \]
where \( D \) is the triangle with vertices \((0,0), (1,1), \) and \((2,0)\).

Solution:

The integral is evaluated as follows:

\[ \int\int_{D} \frac{x}{y+1} \, dA = \int_{0}^{1} \int_{y}^{2-y} \frac{x}{y+1} \, dx \, dy \]
\[ = \int_{0}^{1} \frac{1}{y+1} \left[ \frac{x^2}{2} \right]_{y}^{2-y} \, dy \]
\[ = \int_{0}^{1} \frac{1}{y+1} \left[ \frac{(2-y)^2}{2} - \frac{y^2}{2} \right] \, dy \]
\[ = \frac{1}{2} \int_{0}^{1} \frac{1}{y+1} \left( 4 - 4y + y^2 - y^2 \right) \, dy \]
\[ = \frac{1}{2} \int_{0}^{1} \frac{1}{y+1} (4 - 4y) \, dy \]
\[ = \frac{1}{2} \int_{0}^{1} \frac{1 - y}{1 + y} \, dy \]
\[ = 2 \int_{0}^{1} \left( \frac{2}{1 + y} - 1 \right) \, dy \]
\[ = 2 \left[ 2 \ln(1+y) - y \right]_{0}^{1} \]
\[ = 2 \left[ 2 \ln(1+1) - 1 \right] - 2 \left[ 2 \ln(1+0) - 0 \right] \]
\[ = 4 \ln(2) - 2 \]
17. Let $f(x, y) = x^2 - x + y^2$, and let $\mathcal{D}$ be the bounded region defined by the inequalities $x \geq 0$ and $x \leq 1 - y^2$.

(a) Find and classify the critical points of $f(x, y)$.

(b) Sketch the region $\mathcal{D}$.

(c) Find the absolute maximum and minimum values of $f$ on the region $\mathcal{D}$, and list the points where these values occur.

Solution: First we note that the domain of $f(x, y)$ is bounded and closed, i.e. compact, and that $f(x, y)$ is continuous on the domain. Thus, we are guaranteed to have absolute extrema.

(a) The partial derivatives of $f$ are $f_x = 2x - 1$ and $f_y = 2y$. The critical points of $f$ are all solutions to the system of equations:

\[
\begin{align*}
    f_x &= 2x - 1 = 0 \\
    f_y &= 2y = 0
\end{align*}
\]

The only solution is $x = \frac{1}{2}$ and $y = 0$, which is an interior point of $\mathcal{D}$. The function value at the critical point is:

\[
    f\left(\frac{1}{2}, 0\right) = -\frac{1}{4}
\]

(b) The region $\mathcal{D}$ (shaded) is plotted below along with level curves of $f(x, y)$.
(c) We must now determine the minimum and maximum values of $f$ on the boundary of $\mathcal{D}$. To do this, we must consider each part of the boundary separately:

**Part I** : Let this part be the line segment between $(0, -1)$ and $(0, 1)$. On this part we have $x = 0$ and $-1 \leq y \leq 1$. We now use the fact that $x = 0$ to rewrite $f(x, y)$ as a function of one variable that we call $g_I(y)$.

$$f(x, y) = x^2 - x + y^2$$
$$g_I(y) = y^2$$

The critical points of $g_I(y)$ are:

$$g'_I(y) = 0$$
$$2y = 0$$
$$y = 0$$

Evaluating $g_I(y)$ at the critical point $y = 0$ and at the endpoints of the interval $-1 \leq y \leq 1$, we find that:

$$g_I(0) = 0, \quad g_I(-1) = 1, \quad g_I(1) = 1$$

Note that these correspond to the function values:

$$f(0, 0) = 0, \quad f(0, -1) = 1, \quad f(0, 1) = 1$$

**Part II** : Let this part be the parabola $x = 1 - y^2$ on the interval $-1 \leq y \leq 1$. We now use the fact that $x = 1 - y^2$ to rewrite $f(x, y)$ as a function of one variable that we call $g_{II}(y)$.

$$f(x, y) = x^2 - x + y^2$$
$$g_{II}(y) = (1 - y^2)^2 - (1 - y^2) + y^2$$
$$g_{II}(y) = 1 - 2y^2 + y^4 - 1 + y^2 + y^2$$
$$g_{II}(y) = y^4$$

The critical points of $g_{II}(y)$ are:

$$g'_{II}(y) = 0$$
$$4y^3 = 0$$
$$y = 0$$

Evaluating $g_{II}(y)$ at the critical point $y = 0$ and at the endpoints of the interval $-1 \leq y \leq 1$, we find that:

$$g_{II}(0) = 0, \quad g_{II}(-1) = 1, \quad g_{II}(1) = 1$$

Note that these correspond to the function values:

$$f(1, 0) = 0, \quad f(0, -1) = 1, \quad f(0, 1) = 1$$
Finally, after comparing these values of $f$ we find that the **absolute maximum** of $f$ is 1 at the points $(0, -1)$ and $(0, 1)$ and that the **absolute minimum** of $f$ is $-\frac{1}{4}$ at the point $(\frac{1}{2}, 0)$.

**Note:** In the figure from part (b) we see that the level curves of $f$ are circles centered at $(\frac{1}{2}, 0)$. It is clear that the absolute minimum of $f$ occurs at $(\frac{1}{2}, 0)$ and that the absolute maximum of $f$ occurs at $(0, -1)$ and $(0, 1)$, which are points on the largest circle centered at $(\frac{1}{2}, 0)$ that contains points in $D$. 
18. Consider the function \( F(x, y) = x^2e^{4x-y^2} \). Find the direction (unit vector) in which \( F \) has the fastest growth at the point \((1, 2)\).

**Solution:** The direction in which \( F \) has the fastest growth at the point \((1, 2)\) is the direction of **steepest ascent**:

\[
\hat{u} = \frac{1}{|\nabla F(1, 2)|} \nabla F(1, 2)
\]

The gradient of \( F \) is:

\[
\nabla F = \langle F_x, F_y \rangle
\]

\[
\nabla F = \langle 2xe^{4x-y^2} + 4x^2e^{4x-y^2}, -2xe^{4x-y^2} \rangle
\]

and its value at the point \((1, 2)\) is:

\[
\nabla F(1, 2) = \langle 6, -4 \rangle
\]

Thus, the direction of steepest ascent is:

\[
\hat{u} = \frac{1}{|\nabla F(1, 2)|} \nabla F(1, 2)
\]

\[
= \frac{1}{6, -4} \langle 6, -4 \rangle
\]

\[
= \frac{1}{\sqrt{13}} \langle 3, -2 \rangle
\]
19. Let \( \vec{r}(t) = (e^{-t}, \cos(t)) \) describe movement of a point in the plane, and let \( f(x, y) = x^2y - e^{x+y} \). Use the chain rule to compute the derivative of \( f(\vec{r}(t)) \) at time \( t = 0 \).

**Solution:** We use the Chain Rule for Paths formula:

\[
\frac{d}{dt} f(\vec{r}(t)) = \nabla f \cdot \vec{r}'(t)
\]

where the gradient of \( f \) is:

\[
\nabla f = \langle f_x, f_y \rangle = \langle 2xy - e^{x+y}, x^2 - e^{x+y} \rangle
\]

and the derivative \( \vec{r}'(t) \) is:

\[
\vec{r}'(t) = \langle -e^{-t}, -\sin(t) \rangle
\]

Taking the dot product of these vectors gives us the derivative of \( f(\vec{r}(t)) \).

\[
\frac{d}{dt} f(\vec{r}(t)) = \nabla f \cdot \vec{r}'(t)
\]

\[
\frac{d}{dt} f(\vec{r}(t)) = \langle 2xy - e^{x+y}, x^2 - e^{x+y} \rangle \cdot \langle -e^{-t}, -\sin(t) \rangle
\]

\[
\frac{d}{dt} f(\vec{r}(t)) = -e^{-t} (2xy - e^{x+y}) - \sin(t) (x^2 - e^{x+y})
\]

At \( t = 0 \) we know that \( \vec{r}(0) = (1,1) \) which tells us that \( x = 1 \) and \( y = 1 \). Therefore, plugging \( t = 0, x = 1, \) and \( y = 1 \) into the derivative we find that:

\[
\left. \frac{d}{dt} f(\vec{r}(t)) \right|_{t=0} = -e^{-0} (2(1)(1) - e^{1+1}) - \sin(0) (1^2 - e^{1+1})
\]

\[
\left. \frac{d}{dt} f(\vec{r}(t)) \right|_{t=0} = e^2 - 2
\]
20. Let the function \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) describe the density in the region \( A = \{ x^2 + y^2 + z^2 \leq 1, \sqrt{x^2 + y^2} \leq z \} \). Use spherical coordinates to compute its mass.

**Solution:** The region \( A \) is plotted below.

![Diagram of the region](image)

The mass of the region is given by the triple integral:

\[
\text{mass} = \iiint_A f(x, y, z) \, dV
\]

In Spherical Coordinates, the equations for the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \) are:

- **Sphere:** \( \rho = 1 \)
- **Cone:** \( \phi = \frac{\pi}{4} \)

and the density function \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) is:

- **Density:** \( f(\rho, \phi, \theta) = \rho \)

Using the fact that \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \) in Spherical Coordinates, the mass of the region is:
mass = \iiint_A f(x, y, z) \, dV

= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{1} \rho \left( \rho^2 \sin \phi \right) \, \rho \, d\phi \, d\theta

= \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \left[ \frac{1}{4} \rho^4 \right]_0^1 \, d\phi \, d\theta

= \frac{1}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta

= \frac{1}{4} \int_0^{2\pi} \left[ -\cos \phi \right]_0^{\pi/4} \, d\theta

= \frac{1}{4} \int_0^{2\pi} \left[ -\cos \frac{\pi}{4} - (-\cos 0) \right] \, d\theta

= \frac{1}{4} \int_0^{2\pi} \left( -\frac{\sqrt{2}}{2} + 1 \right) \, d\theta

= \frac{1}{4} \left( 1 - \frac{\sqrt{2}}{2} \right) \left[ \theta \right]_0^{2\pi}

= \frac{\pi}{2} \left( 1 - \frac{\sqrt{2}}{2} \right)