1. Find and classify the critical points of the function

\[ f(x, y) = x^3 + 3xy - y^3. \]

**Solution:** By definition, an interior point \((a, b)\) in the domain of \(f\) is a **critical point** of \(f\) if either

1. \(f_x(a, b) = f_y(a, b) = 0\), or
2. one (or both) of \(f_x\) or \(f_y\) does not exist at \((a, b)\).

The partial derivatives of \(f(x, y) = x^3 + 3xy - y^3\) are \(f_x = 3x^2 + 3y\) and \(f_y = 3x - 3y^2\). These derivatives exist for all \((x, y)\) in \(\mathbb{R}^2\). Thus, the critical points of \(f\) are the solutions to the system of equations:

\[
\begin{align*}
    f_x &= 3x^2 + 3y = 0 \quad (1) \\
    f_y &= 3x - 3y^2 = 0 \quad (2)
\end{align*}
\]

Solving Equation (1) for \(y\) we get:

\[ y = -x^2 \quad (3) \]

Substituting this into Equation (2) and solving for \(x\) we get:

\[
\begin{align*}
    3x - 3y^2 &= 0 \\
    3x - 3(-x^2)^2 &= 0 \\
    3x - 3x^4 &= 0 \\
    3x(1 - x^3) &= 0
\end{align*}
\]

We observe that the above equation is satisfied if either \(x = 0\) or \(x^3 - 1 = 0 \iff x = 1\). We find the corresponding \(y\)-values using Equation (3): \(y = -x^2\).

- If \(x = 0\), then \(y = -0^2 = 0\).
- If \(x = 1\), then \(y = -(1)^2 = -1\).

Thus, the critical points are \(\{(0, 0)\} \) and \(\{(1, -1)\} \).

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of \(f\) are:

\[ f_{xx} = 6x, \quad f_{yy} = -6y, \quad f_{xy} = 3 \]
The discriminant function $D(x, y)$ is then:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2$$

$$D(x, y) = (6x)(-6y) - (3)^2$$

$$D(x, y) = -36xy - 9$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

<table>
<thead>
<tr>
<th>$(a, b)$</th>
<th>$D(a, b)$</th>
<th>$f_{xx}(a, b)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>$-9$</td>
<td>$0$</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>$(1, -1)$</td>
<td>$27$</td>
<td>$6$</td>
<td>Local Minimum</td>
</tr>
</tbody>
</table>

Recall that $(a, b)$ is a saddle point if $D(a, b) < 0$ and that $(a, b)$ corresponds to a local minimum of $f$ if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$.

Figure 1: Pictured above are level curves of $f(x, y)$. Darker colors correspond to smaller values of $f(x, y)$. It is apparent that $(0, 0)$ is a saddle point and $(1, -1)$ corresponds to a local minimum.
2. Sketch the region of integration and compute $\int_0^1 \int_0^y e^{-y^2} \, dx \, dy$.

**Solution:**

The integral is evaluated as follows:

\[
\int_0^1 \int_0^y e^{-y^2} \, dx \, dy = \int_0^2 \left[ xe^{-y^2} \right]_0^y \, dy
\]

\[
= \int_0^2 ye^{-y^2} \, dy
\]

\[
= \left[ -\frac{1}{2} e^{-y^2} \right]_0^1
\]

\[
= \left[ -\frac{1}{2} e^{-1} \right] - \left[ -\frac{1}{2} e^0 \right]
\]

\[
= \frac{1}{2} - \frac{1}{2} e^{-1}
\]
3. Compute

\[ \iiint_{A} z \, dV \]

where \( A \) is the region inside the sphere \( x^2 + y^2 + z^2 = 2 \), inside the cylinder \( x^2 + y^2 = 1 \), and above the \( xy \)-plane.

Solution: The region \( A \) is plotted below.

We use Cylindrical Coordinates to evaluate the triple integral. The equations for the sphere and cylinder are then:

\[
\begin{align*}
\text{Sphere:} & \quad r^2 + z^2 = 2 \quad \Rightarrow \quad z = \sqrt{2 - r^2} \\
\text{Cylinder:} & \quad r^2 = 1 \quad \Rightarrow \quad r = 1
\end{align*}
\]

The surface that bounds \( A \) from below is \( z = 0 \) (the \( xy \)-plane) and the surface that bounds \( A \) from above is \( z = \sqrt{2 - r^2} \) (the sphere). The projection of the region \( A \) onto the \( xy \)-plane is the disk \( 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi \). Using the fact that \( dV = r \, dz \, dr \, d\theta \) in Cylindrical Coordinates, the value of the triple integral is:
\begin{align*}
\iiint_{A} z \, dV &= \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{\sqrt{2-r^2}} zr \, dz \, dr \, d\theta \\
&= \int_{0}^{2\pi} \int_{0}^{1} r \left[ \frac{1}{2} z^2 \right]_{0}^{\sqrt{2-r^2}} \, dr \, d\theta \\
&= \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{1} r (2 - r^2) \, dr \, d\theta \\
&= \frac{1}{2} \int_{0}^{2\pi} \left[ r^2 - \frac{1}{4} r^4 \right]_{0}^{1} \, d\theta \\
&= \frac{1}{2} \int_{0}^{2\pi} \frac{3}{4} \, d\theta \\
&= \frac{3 \theta}{8} \bigg|_{0}^{2\pi} \\
&= \frac{3\pi}{4}
\end{align*}
4. Compute the integral of the field $\mathbf{F}(x, y) = (x + y) \mathbf{i} + 0 \mathbf{j}$ along the curve $\mathbf{C}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$.

Solution: By definition, the line integral of a vector field $\mathbf{F}$ along a curve $C$ with parameterization $\mathbf{C}(\theta) = \langle x(\theta), y(\theta) \rangle$, $a \leq \theta \leq b$ is given by the formula:

$$\int_{C} \mathbf{F} \cdot \mathbf{T} \, ds = \int_{a}^{b} \mathbf{F} \cdot \mathbf{C}'(\theta) \, d\theta$$

From the given parameterization $\mathbf{C}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ we have:

$$\mathbf{C}'(\theta) = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$$

and, using the fact that $x(\theta) = \cos \theta$ and $y(\theta) = \sin \theta$, the function $\mathbf{F}$ can be rewritten as:

$$\mathbf{F} = (x + y) \mathbf{i} + 0 \mathbf{j}$$
$$\mathbf{F} = (\cos \theta + \sin \theta) \mathbf{i} + 0 \mathbf{j}$$

Assuming an interval $0 \leq \theta \leq 2\pi$, the value of the line integral is then:

$$\int_{C} \mathbf{F} \cdot \mathbf{T} \, ds = \int_{a}^{b} \mathbf{F} \cdot \mathbf{C}'(\theta) \, d\theta$$
$$= \int_{0}^{2\pi} ((\cos \theta + \sin \theta) \mathbf{i} + 0 \mathbf{j}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \, d\theta$$
$$= \int_{0}^{2\pi} (\cos \theta + \sin \theta)(-\sin \theta) \, d\theta$$
$$= \int_{0}^{2\pi} (-\sin \theta \cos \theta - \sin^2 \theta) \, d\theta$$
$$= \left[ \frac{1}{2} \cos^2 \theta - \frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) \right]_{0}^{2\pi}$$
$$= \left[ \frac{1}{2} \cos^2(2\pi) - \frac{1}{2} (2\pi) + \frac{1}{4} \sin(4\pi) \right] - \left[ \frac{1}{2} \cos^2 0 - \frac{1}{2} (0) + \frac{1}{4} \sin(0) \right]$$
$$= -\pi$$
5. Find the minimum and maximum of the function \( f(x, y) = x + y^2 \) subject to the condition \( 2x^2 + y^2 = 1 \).

**Solution:** We find the minimum and maximum using the method of Lagrange Multipliers. First, we recognize that \( 2x^2 + y^2 = 1 \) is compact which guarantees the existence of absolute extrema of \( f \). Then, let \( g(x, y) = 2x^2 + y^2 = 1 \). We look for solutions to the following system of equations:

\[
\begin{align*}
    f_x &= \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 1
\end{align*}
\]

which, when applied to our functions \( f \) and \( g \), give us:

\[
\begin{align*}
    1 &= \lambda (4x) \\
    2y &= \lambda (2y) \\
    2x^2 + y^2 &= 1
\end{align*}
\]

We begin by noting that Equation (2) gives us:

\[
\begin{align*}
    2y &= \lambda (2y) \\
    2y - \lambda (2y) &= 0 \\
    2y(1 - \lambda) &= 0
\end{align*}
\]

From this equation we either have \( y = 0 \) or \( \lambda = 1 \). Let’s consider each case separately.

**Case 1:** Let \( y = 0 \). We find the corresponding \( x \)-values using Equation (3).

\[
\begin{align*}
    2x^2 + y^2 &= 1 \\
    2x^2 + 0^2 &= 1 \\
    x^2 &= \frac{1}{2} \\
    x &= \pm \frac{1}{\sqrt{2}}
\end{align*}
\]

Thus, the points of interest are \( \left( \frac{1}{\sqrt{2}}, 0 \right) \) and \( \left( -\frac{1}{\sqrt{2}}, 0 \right) \).

**Case 2:** Let \( \lambda = 1 \). Plugging this into Equation (1) we get:

\[
\begin{align*}
    1 &= \lambda (4x) \\
    1 &= 1(4x) \\
    x &= \frac{1}{4}
\end{align*}
\]

We find the corresponding \( y \)-values using Equation (3).

\[
\begin{align*}
    2x^2 + y^2 &= 1 \\
    2 \left( \frac{1}{4} \right)^2 + y^2 &= 1 \\
    \frac{1}{8} + y^2 &= 1 \\
    y^2 &= \frac{7}{8} \\
    y &= \pm \sqrt{\frac{7}{8}}
\end{align*}
\]
Thus, the points of interest are \((\frac{1}{4}, \sqrt{\frac{2}{8}})\) and \((\frac{1}{4}, -\sqrt{\frac{2}{8}})\).

We now evaluate \(f(x, y) = x + y^2\) at each point of interest obtained by Cases 1 and 2.

\[
\begin{align*}
  f\left(\frac{1}{\sqrt{2}}, 0\right) & = \frac{1}{\sqrt{2}} \\
  f\left(-\frac{1}{\sqrt{2}}, 0\right) & = -\frac{1}{\sqrt{2}} \\
  f\left(\frac{1}{4}, \sqrt{\frac{2}{8}}\right) & = \frac{9}{8} \\
  f\left(\frac{1}{4}, -\sqrt{\frac{2}{8}}\right) & = \frac{9}{8}
\end{align*}
\]

From the values above we observe that \(f\) attains an absolute maximum of \(\frac{9}{8}\) and an absolute minimum of \(-\frac{1}{\sqrt{2}}\).

Figure 1: Shown in the figure are the level curves of \(f(x, y) = x + y^2\) and the ellipse \(2x^2 + y^2 = 1\) (thick, black curve). Darker colors correspond to smaller values of \(f(x, y)\). Notice that (1) the parabola \(f(x, y) = x + y^2 = \frac{9}{8}\) is tangent to the ellipse at the points \((\frac{1}{4}, \sqrt{\frac{2}{8}})\) and \((\frac{1}{4}, -\sqrt{\frac{2}{8}})\) which correspond to the absolute maximum and (2) the parabola \(f(x, y) = x + y^2 = -\frac{1}{\sqrt{2}}\) is tangent to the ellipse at the point \((-\frac{1}{\sqrt{2}}, 0)\) which corresponds to the absolute minimum.
Math 210, Exam 2, Spring 2010
Problem 6 Solution

6. For the vector field $\mathbf{F}(x, y) = (x + y) \mathbf{i} + (x - y) \mathbf{j}$, find a function $\varphi(x, y)$ with $\text{grad} \varphi = \mathbf{F}$ or use the partial derivative test to show that such a function does not exist.

**Solution:** In order for the vector field $\mathbf{F} = (f(x, y), g(x, y))$ to have a potential function $\varphi(x, y)$ such that $\text{grad} \varphi = \mathbf{F}$, it must be the case that:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Using $f(x, y) = x + y$ and $g(x, y) = x - y$ we get:

$$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = 1$$

which verifies the existence of a potential function for the given vector field.

If $\text{grad} \varphi = \mathbf{F}$, then it must be the case that:

$$\frac{\partial \varphi}{\partial x} = f(x, y) \quad (1)$$
$$\frac{\partial \varphi}{\partial y} = g(x, y) \quad (2)$$

Using $f(x, y) = x + y$ and integrating both sides of Equation (1) with respect to $x$ we get:

$$\frac{\partial \varphi}{\partial x} = f(x, y)$$
$$\frac{\partial \varphi}{\partial x} = x + y$$

$$\int \frac{\partial \varphi}{\partial x} \, dx = \int (x + y) \, dx$$

$$\varphi(x, y) = \frac{1}{2} x^2 + xy + h(y) \quad (3)$$

We obtain the function $h(y)$ using Equation (2). Using $g(x, y) = x - y$ we get the equation:

$$\frac{\partial \varphi}{\partial y} = g(x, y)$$
$$\frac{\partial \varphi}{\partial y} = x - y$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\frac{\partial}{\partial y} \left( \frac{1}{2} x^2 + xy + h(y) \right) = x - y$$

$$x + h'(y) = x - y$$

$$h'(y) = -y$$

1
Now integrate both sides with respect to $y$ to get:

\[
\int h'(y) \, dy = \int -y \, dy
\]

\[
h(y) = -\frac{1}{2} y^2 + C
\]

Letting $C = 0$, we find that a potential function for $\mathbf{F}$ is:

\[
\varphi(x, y) = \frac{1}{2} x^2 + xy - \frac{1}{2} y^2
\]