1. Consider the vector field \( \mathbf{F} = \langle y^2, 2xy + 2y \rangle \).

(a) Show that \( \mathbf{F} \) is conservative.

(b) Find a potential function \( \varphi \) such that \( \mathbf{F} = \nabla \varphi \).

(c) Compute \( \int_C \mathbf{F} \cdot d\mathbf{s} \) along any path \( C \) from \((-1, 2)\) to \((3, 0)\).

Solution:

(a) In order for the vector field \( \mathbf{F} = \langle f(x, y), g(x, y) \rangle \) to be conservative, it must be the case that:
\[
\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}
\]

Using \( f(x, y) = y^2 \) and \( g(x, y) = 2xy + 2y \) we get:
\[
\frac{\partial f}{\partial y} = 2y, \quad \frac{\partial g}{\partial x} = 2y
\]

Thus, the vector field is conservative.

(b) If \( \mathbf{F} = \nabla \varphi \), then it must be the case that:
\[
\frac{\partial \varphi}{\partial x} = f(x, y) \quad (1)
\]
\[
\frac{\partial \varphi}{\partial y} = g(x, y) \quad (2)
\]

Using \( f(x, y) = y^2 \) and integrating both sides of Equation (1) with respect to \( x \) we get:
\[
\frac{\partial \varphi}{\partial x} = f(x, y)
\]
\[
\frac{\partial \varphi}{\partial y} = y^2
\]
\[
\int \frac{\partial \varphi}{\partial x} \, dx = \int (y^2) \, dx
\]
\[
\varphi(x, y) = xy^2 + h(y) \quad (3)
\]

We obtain the function \( h(y) \) using Equation (2). Using \( g(x, y) = 2xy + 2y \) we get the equation:
\[
\frac{\partial \varphi}{\partial y} = g(x, y)
\]
\[
\frac{\partial \varphi}{\partial y} = 2xy + 2y
\]
We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

\[
\frac{\partial}{\partial y} \left( xy^2 + h(y) \right) = 2xy + 2y
\]

\[
2xy + h'(y) = 2xy + 2y
\]

\[
h'(y) = 2y
\]

Now integrate both sides with respect to \( y \) to get:

\[
\int h'(y) \, dy = \int 2y \, dy
\]

\[
h(y) = y^2 + C
\]

Letting \( C = 0 \), we find that a potential function for \( \vec{F} \) is:

\[
\varphi(x, y) = xy^2 + y^2
\]

(c) Using the Fundamental Theorem of Line Integrals, we have:

\[
\int_C \vec{F} \cdot d\vec{s} = \varphi(3, 0) - \varphi(-1, 2)
\]

\[
= [3(0)^2 + 0^2] - [(-1)(2)^2 + 2^2]
\]

\[
= 0
\]
2. Complete each of the following:

(a) Consider a particle whose position vector is given by:
\[ \mathbf{r}(t) = \langle \sin(\pi t), t^2, t + 1 \rangle \]

Find the velocity, speed, and acceleration of the particle at \( t = 2 \).

(b) Find the directional derivative \( D_\mathbf{u}f \) of the function \( f(x, y) = e^{x+y} \sin(xy) \) at the point \((\pi, 1)\) in the direction of \( \nabla = \langle 4, 0 \rangle \).

Solution:

(a) The velocity, speed, and acceleration functions are:
\[ \nabla(t) = \mathbf{r}'(t) = \langle \pi \cos(\pi t), 2t, 1 \rangle \]
\[ v(t) = ||\nabla(t)|| = \sqrt{\pi^2 \cos^2(\pi t) + 4t^2 + 1} \]
\[ \mathbf{a}(t) = \nabla'(t) = \langle -\pi^2 \sin(\pi t), 2, 0 \rangle \]

At \( t = 2 \) we have:
\[ \nabla(2) = \langle \pi, 4, 1 \rangle \]
\[ v(t) = \sqrt{\pi^2 + 17} \]
\[ \mathbf{a}(t) = \langle 0, 2, 0 \rangle \]

(b) The directional derivative of \( f(x, y) \) at \((a, b)\) in the direction of the unit vector \( \hat{\mathbf{u}} \) is, by definition:
\[ D_\mathbf{u}f(a, b) = \nabla f(a, b) \cdot \hat{\mathbf{u}} \]

The gradient of \( f(x, y) \) is:
\[ \nabla f = \langle f_x, f_y \rangle = \langle e^{x+y}(\sin(xy) + y \cos(xy)), e^{x+y}(\sin(xy) + x \cos(xy)) \rangle \]

At the point \((\pi, 1)\) we have:
\[ \nabla f(\pi, 1) = \langle -e^{\pi+1}, -\pi e^{\pi+1} \rangle \]

The unit vector \( \hat{\mathbf{u}} \) in the direction of \( \nabla = \langle 4, 0 \rangle \) is:
\[ \hat{\mathbf{u}} = \frac{1}{||\nabla||} \nabla = \frac{1}{4} \langle 4, 0 \rangle = \langle 1, 0 \rangle \]

Thus, the directional derivative is:
\[ D_\mathbf{u}f(\pi, 1) = \nabla f(\pi, 1) \cdot \hat{\mathbf{u}} \]
\[ = \langle -e^{\pi+1}, -\pi e^{\pi+1} \rangle \cdot \langle 1, 0 \rangle \]
\[ = -e^{\pi+1} \]
3. Use a triple integral to compute the volume of the region below the sphere \( x^2 + y^2 + z^2 = 4 \) and above the disk \( x^2 + y^2 \leq 1 \) in the \( xy \)-plane. (Hint: Use cylindrical coordinates.)

Solution: The region of integration is shown below.

The equation for the sphere in cylindrical coordinates is \( r^2 + z^2 = 4 \implies z = \sqrt{4 - r^2} \) since the region is above the \( xy \)-plane. Furthermore, the disk in the \( xy \)-plane is described by \( 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi \) in cylindrical coordinates. Thus, the volume of the region is:

\[
V = \iiint_R 1 \, dV \\
= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} 1 \, r \, dz \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^1 r \sqrt{4-r^2} \, dr \, d\theta \\
= \int_0^{2\pi} \left[ -\frac{1}{3} (4-r^2)^{3/2} \right]_0^1 d\theta \\
= \int_0^{2\pi} \left[ -\frac{1}{3} (4-1^2)^{3/2} + \frac{1}{3} (4-0^2)^{3/2} \right] d\theta \\
= \int_0^{2\pi} \frac{1}{3} (8-3\sqrt{3}) \, d\theta \\
= \frac{2\pi}{3} \left( 8-3\sqrt{3} \right)
\]
4. Find the critical points of the function \( f(x, y) = x^2 + y^2 + x^2y + 1 \) and classify each point as corresponding to either a saddle point, a local minimum, or a local maximum.

**Solution:** By definition, an interior point \((a, b)\) in the domain of \(f\) is a critical point of \(f\) if either

1. \(f_x(a, b) = f_y(a, b) = 0\), or
2. one (or both) of \(f_x\) or \(f_y\) does not exist at \((a, b)\).

The partial derivatives of \(f(x, y) = x^2 + y^2 + x^2y + 1\) are \(f_x = 2x + 2xy\) and \(f_y = 2y + x^2\). These derivatives exist for all \((x, y)\) in \(\mathbb{R}^2\). Thus, the critical points of \(f\) are the solutions to the system of equations:

\[
\begin{align*}
  f_x &= 2x + 2xy = 0 \\
  f_y &= 2y + x^2 = 0
\end{align*}
\]

Factoring Equation (1) gives us:

\[
2x + 2xy = 0 \\
2x(1 + y) = 0
\]

\(x = 0\), or \(y = -1\)

If \(x = 0\) then Equation (2) gives us \(y = 0\). If \(y = -1\) then Equation (2) gives us:

\[
2(-1) + x^2 = 0 \\
x^2 = 2 \\
x = \pm \sqrt{2}
\]

Thus, the critical points are \((0, 0)\), \((\sqrt{2}, -1)\), and \((-\sqrt{2}, -1)\).

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of \(f\) are:

\[
\begin{align*}
  f_{xx} &= 2 + 2y, \\
  f_{yy} &= 2, \\
  f_{xy} &= 2
\end{align*}
\]

The discriminant function \(D(x, y)\) is then:

\[
\begin{align*}
  D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\
  D(x, y) &= (2 + 2y)(2) - (2x)^2 \\
  D(x, y) &= 4 + 4y - 4x^2
\end{align*}
\]

The values of \(D(x, y)\) at the critical points and the conclusions of the Second Derivative Test are shown in the table below.
Recall that \((a, b)\) is a saddle point if \(D(a, b) < 0\) and that \((a, b)\) corresponds to a local minimum of \(f\) if \(D(a, b) > 0\) and \(f_{xx}(a, b) > 0\).
5. Compute the integral $\int\int_D (x+3) \, dA$ where $D$ is the region bounded by the curves $y = 1-x$ and $y = 1-x^2$.

**Solution:**

From the figure we see that the region $D$ is bounded above by $y = 1-x^2$ and below by $y = 1-x$. The projection of $D$ onto the $x$-axis is the interval $0 \leq x \leq 1$. Using the order of integration $dy \, dx$ we have:

$$\int\int_D (x+3) \, dA = \int_0^1 \int_{1-x}^{1-x^2} (x+3) \, dy \, dx$$

$$= \int_0^1 (x+3) \left[ y \right]_{1-x}^{1-x^2} \, dx$$

$$= \int_0^1 (x+3)[(1-x^2) - (1-x)] \, dx$$

$$= \int_0^1 (x+3)(x-x^2) \, dx$$

$$= \int_0^1 (-x^3 - 2x^2 + 3x) \, dx$$

$$= \left[ -\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 \right]_0^1$$

$$= \frac{7}{12}$$
6. Let $S$ be the portion of the plane $x + y + z = 6$ that lies above the square $0 \leq x \leq 2$, $1 \leq y \leq 3$ in the $xy$ plane. Compute the integral $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, y, z \rangle$ and the normal vector to $S$ points upward.

**Solution:** The formula we will use to compute the surface integral of the vector field $\vec{F}$ is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (\vec{T}_u \times \vec{T}_v) \ dA$$

where the function $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ with domain $R$ is a parameterization of the surface $S$ and the vectors $\vec{T}_u = \frac{\partial \vec{r}}{\partial u}$ and $\vec{T}_v = \frac{\partial \vec{r}}{\partial v}$ are the tangent vectors.

We begin by finding a parameterization of the plane. Let $x = u$ and $y = v$. Then, $z = 6 - x - y$ using the equation for the plane. Thus, we have $\vec{r}(u, v) = \langle u, v, 6 - u - v \rangle$. Furthermore, the domain $R$ is the set of all points $(u, v)$ satisfying $0 \leq u \leq 2$ and $1 \leq v \leq 3$. Therefore, a parameterization of $S$ is:

$$\vec{r}(u, v) = \langle u, v, 6 - u - v \rangle,$$

$$R = \left\{ (u, v) \mid 0 \leq u \leq 2, 1 \leq v \leq 3 \right\}$$

The tangent vectors $\vec{T}_u$ and $\vec{T}_v$ are then:

$$\vec{T}_u = \frac{\partial \vec{r}}{\partial u} = \langle 1, 0, -1 \rangle$$

$$\vec{T}_v = \frac{\partial \vec{r}}{\partial v} = \langle 0, 1, -1 \rangle$$

The cross product of these vectors is:

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

The vector field $\vec{F} = \langle x, y, z \rangle$ written in terms of $u$ and $v$ is:

$$\vec{F} = \langle u, v, 6 - u - v \rangle$$

Before computing the surface integral, we note that $\vec{T}_u \times \vec{T}_v$ points upward, as desired, since the third component of the vector is positive.
The value of the surface integral is:

\[ \int \int_S \vec{F} \cdot d\vec{S} = \int \int_R \vec{F} \cdot (\vec{T}_u \times \vec{T}_v) \ dA \]

\[ = \int \int_R \langle u, v, 6 - u - v \rangle \cdot \langle 1, 1, 1 \rangle \ dA \]

\[ = \int \int_R (u + v + 6 - u - v) \ dA \]

\[ = \int \int_R 6 \ dA \]

\[ = 6 \int \int_R 1 \ dA \]

\[ = 6 \times (\text{Area of } R) \]

\[ = 6 \times 4 \]

\[ = 24 \]
7. Find an equation for the plane that contains the points \((1, 2, 1), (-3, 0, 1),\) and \((2, 2, 0).\)

**Solution:** A vector \(\vec{n}\) perpendicular to the plane is the cross product of any two non-parallel vectors that lie in the plane. Let \(\vec{u} = \langle -3 - 1, 0 - 2, 1 - 1 \rangle = \langle -4, -2, 0 \rangle\) be the vector from \((1, 2, 1)\) to \((-3, 0, 1)\) and \(\vec{v} = \langle 2 - 1, 2 - 2, 0 - 1 \rangle = \langle 1, 0, -1 \rangle\) be the vector from \((1, 2, 1)\) to \((2, 2, 0).\) Then the normal vector is:

\[
\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix}
i & j & \hat{k} \\
-4 & -2 & 0 \\
1 & 0 & -1
\end{vmatrix} = \hat{i} \begin{vmatrix}
-2 & 0 \\
0 & -1
\end{vmatrix} - \hat{j} \begin{vmatrix}
-4 & 0 \\
1 & -1
\end{vmatrix} + \hat{k} \begin{vmatrix}
-4 & -2 \\
1 & 0
\end{vmatrix} = 2\hat{i} - 4\hat{j} + 2\hat{k} = \langle 2, -4, 2 \rangle
\]

Using \((1, 2, 1)\) as a point on the plane, we have:

\[
2(x - 1) - 4(y - 2) + 2(z - 1) = 0
\]
8. Consider the vector field \( \mathbf{F} = (e^{2x} + y, 4x + \sin(y^2)) \) and the curve \( C \) shown below. Use Green’s Theorem to compute \( \oint_C \mathbf{F} \cdot d\mathbf{s} \). (Note: Each square in the grid has a side of length \( \frac{1}{4} \).)

**Solution:** Green’s Theorem states that
\[
\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA
\]
where \( D \) is the region enclosed by \( C \). The integrand of the double integral is:
\[
\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \frac{\partial}{\partial x} (4x + \sin(y^2)) - \frac{\partial}{\partial y} (e^{2x} + y)
\]
\[
= 4 - 1
\]
\[
= 3
\]
Thus, the value of the integral is:
\[
\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA
\]
\[
= \iint_D 3\,dA
\]
\[
= 3 \int \int_D 1\,dA
\]
\[
= 3 \times \text{(area of } D)\]
\[
= 3 \times 4
\]
\[
= 12
\]
Note that $D$ consists of 64 squares and each has an area of $(\frac{1}{4})^2 = \frac{1}{16}$ so the area of $D$ is $64 \times \frac{1}{16} = 4$. 