1. The position vector
\[ \vec{r}(t) = t^3 \hat{i} + 18t \hat{j} + 3t^{-1} \hat{k}, \quad 1 \leq t \leq 2 \]
describes the motion of a particle.
(a) Find the position at time \( t = 2 \).
(b) Find the velocity at time \( t = 2 \).
(c) Find the acceleration at time \( t = 2 \).
(d) Find the length of the path traveled by the particle during the time \( 1 \leq t \leq 2 \).

Solution:
(a) The position at time \( t = 2 \) is:
\[ \vec{r}(2) = 2^3 \hat{i} + 18(2) \hat{j} + 3(2)^{-1} \hat{k} = 8 \hat{i} + 36 \hat{j} + \frac{3}{2} \hat{k} \]

(b) The velocity is the derivative of position.
\[ \vec{v}(t) = \vec{r}'(t) = 3t^2 \hat{i} + 18 \hat{j} - 3t^{-2} \hat{k} \]
Therefore, the velocity at time \( t = 2 \) is:
\[ \vec{v}(2) = 3(2)^2 \hat{i} + 18 \hat{j} - 3(2)^{-2} \hat{k} = 12 \hat{i} + 18 \hat{j} - \frac{3}{4} \hat{k} \]

(c) The acceleration is the derivative of velocity.
\[ \vec{a}(t) = \vec{v}'(t) = 6t \hat{i} + 6t^{-3} \hat{k} \]
Therefore, the acceleration at time \( t = 2 \) is:
\[ \vec{a}(2) = 6(2) \hat{i} + 6(2)^{-3} \hat{k} = 12 \hat{i} + \frac{3}{4} \hat{k} \]

(d) The length of the path traveled by the particle is:
\[ L = \int_1^2 \| \vec{r}'(t) \| \, dt \]
\[ = \int_1^2 \sqrt{(3t^2)^2 + 18^2 + (-3t^{-2})^2} \, dt \]
\[ = \int_1^2 \sqrt{9t^4 + 324 + 9t^{-4}} \, dt \]

It turns out that a simple antiderivative of the integrand does not exist. There was a typo in the original problem. The \( \hat{j} \)-component of \( \vec{r}(t) \) should have been \( \sqrt{18t} \) not \( 18t \).
2. (a) For $f(x, y) = e^{(x+1)y}$ find the derivatives:

$$
\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}
$$

(b) Find the gradient of $f$ at the point $(2, 3)$.

Solution:

(a) The first partial derivatives of $f(x, y)$ are

$$
\frac{\partial f}{\partial x} = ye^{(x+1)y}
$$

$$
\frac{\partial f}{\partial y} = (x + 1)e^{(x+1)y}
$$

The second derivatives are:

$$
\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x} \left( ye^{(x+1)y} \right) = y^2 e^{(x+1)y}
$$

$$
\frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y} \left( (x + 1)e^{(x+1)y} \right) = (x + 1)^2 e^{(x+1)y}
$$

$$
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( (x + 1)e^{(x+1)y} \right) = e^{(x+1)y} + y(x + 1)e^{(x+1)y}
$$

(b) The gradient of $f$ at $(2, 3)$ is:

$$
\nabla f(2, 3) = \langle f_x(2, 3), f_y(2, 3) \rangle
$$

$$
= \langle 3e^{(2+1)^3}, (2 + 1)e^{(2+1)^3} \rangle
$$

$$
= \langle 3e^9, 3e^9 \rangle
$$
3. (a) Find a potential function for the vector field

\[ \vec{F}(x, y, z) = (1 - z) \hat{i} + y \hat{j} - x \hat{k} \]

(b) Integrate \( \vec{F} \) over the straight line from \((1, 0, 1)\) to \((0, 1, 2)\).

[You may calculate this directly or you may use a potential function.]

Solution:

(a) By inspection, a potential function for the vector field \( \vec{F} \) is:

\[ \varphi(x, y, z) = x - xz + \frac{1}{2} y^2 \]

To verify, we calculate the gradient of \( \varphi \):

\[
\nabla \varphi = \varphi_x \hat{i} + \varphi_y \hat{j} + \varphi_z \hat{k}
\]

\[= (1 - z) \hat{i} + y \hat{j} - x \hat{k} = \vec{F} \]

(b) Using the Fundamental Theorem of Line Integrals, the value of the line integral is:

\[
\int_C \vec{F} \cdot d\vec{s} = \varphi(0, 1, 2) - \varphi(1, 0, 1)
\]

\[= \left[ 0 - (0)(2) + \frac{1}{2}(1)^2 \right] - \left[ 1 - (1)(1) + \frac{1}{2}(0)^2 \right]
\]

\[= \frac{1}{2} \]
4. (a) Find the critical points of the function \( f(x, y) = x^3 - 3x - y^2 \).

(b) Use the second derivative test to classify each critical point as a local maximum, local minimum, or saddle.

Solution:

(a) By definition, an interior point \((a, b)\) in the domain of \( f \) is a critical point of \( f \) if either

1. \( f_x(a, b) = f_y(a, b) = 0 \), or
2. one (or both) of \( f_x \) or \( f_y \) does not exist at \((a, b)\).

The partial derivatives of \( f(x, y) = x^3 - 3x - y^2 \) are \( f_x = 3x^2 - 3 \) and \( f_y = -2y \). These derivatives exist for all \((x, y)\) in \( \mathbb{R}^2 \). Thus, the critical points of \( f \) are the solutions to the system of equations:

\[
\begin{align*}
  f_x &= 3x^2 - 3 = 0 \quad (1) \\
  f_y &= -2y = 0 \quad (2)
\end{align*}
\]

The two solutions to Equation (1) are \( x = \pm 1 \). The only solution to Equation (2) is \( y = 0 \). Thus, the critical points are \((1, 0)\) and \((-1, 0)\).

(b) We now use the Second Derivative Test to classify the critical points. The second derivatives of \( f \) are:

\[
\begin{align*}
  f_{xx} &= 6x, \quad f_{yy} = -2, \quad f_{xy} = 0
\end{align*}
\]

The discriminant function \( D(x, y) \) is then:

\[
\begin{align*}
  D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\
  D(x, y) &= (6x)(-2) - (0)^2 \\
  D(x, y) &= -12x
\end{align*}
\]

The values of \( D(x, y) \) at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

<table>
<thead>
<tr>
<th>((a, b))</th>
<th>(D(a, b))</th>
<th>(f_{xx}(a, b))</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 0))</td>
<td>-12</td>
<td>6</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>((-1, 0))</td>
<td>12</td>
<td>-6</td>
<td>Local Maximum</td>
</tr>
</tbody>
</table>

Recall that \((a, b)\) is a saddle point if \( D(a, b) < 0 \) and that \((a, b)\) corresponds to a local maximum of \( f \) if \( D(a, b) > 0 \) and \( f_{xx}(a, b) < 0 \).
5. Find the maximum and minimum of the function \( f(x, y) = (x - 1)^2 + y^2 \) subject to the constraint:

\[
g(x, y) = \left( \frac{x}{3} \right)^2 + \left( \frac{y}{2} \right)^2 = 1
\]

**Solution:** We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that \((\frac{x}{3})^2 + (\frac{y}{2})^2 = 1\) is compact which guarantees the existence of absolute extrema of \( f \). We look for solutions to the following system of equations:

\[
\begin{align*}
2(x - 1) &= \lambda \left( \frac{2x}{9} \right) \quad (1) \\
2y &= \lambda \left( \frac{y}{2} \right) \quad (2) \\
\left( \frac{x}{3} \right)^2 + \left( \frac{y}{2} \right)^2 &= 1 \quad (3)
\end{align*}
\]

From Equation (2) we observe that:

\[
\begin{align*}
2y &= \lambda \left( \frac{y}{2} \right) \\
4y &= \lambda y \\
4y - \lambda y &= 0 \\
y(4 - \lambda) &= 0 \\
y = 0, \; \text{or} \; \lambda = 4
\end{align*}
\]

If \( y = 0 \) then Equation (3) gives us:

\[
\left( \frac{x}{3} \right)^2 + \left( \frac{0}{2} \right)^2 = 1
\]

\[
\frac{x^2}{9} = 1 \\
x^2 = 9 \\
x = \pm 3
\]
If $\lambda = 4$ then Equation (1) gives us:

$$2(x - 1) = \lambda \left( \frac{2x}{9} \right)$$
$$2(x - 1) = 4 \left( \frac{2x}{9} \right)$$

$$x - 1 = \frac{4x}{9}$$
$$\frac{5x}{9} = 1$$

$$x = \frac{9}{5}$$

which, when plugged into Equation (3), gives us:

$$\left( \frac{x}{3} \right)^2 + \left( \frac{y}{2} \right)^2 = 1$$
$$\left( \frac{9/5}{3} \right)^2 + \frac{y^2}{4} = 1$$
$$\frac{9}{25} + \frac{y^2}{4} = 1$$
$$\frac{y^2}{4} = \frac{16}{25}$$
$$y^2 = \frac{64}{25}$$
$$y = \pm \frac{8}{5}$$

Thus, the points of interest are $(3, 0), (-3, 0), \left( \frac{9}{5}, \frac{8}{5} \right)$, and $\left( \frac{9}{5}, -\frac{8}{5} \right)$.

We now evaluate $f(x, y) = (x - 1)^2 + y^2$ at each point of interest.

$$f(3, 0) = (3 - 1)^2 + 0^2 = 4$$
$$f(-3, 0) = (-3 - 1)^2 + 0^2 = 16$$
$$f\left( \frac{9}{5}, \frac{8}{5} \right) = \left( \frac{9}{5} - 1 \right)^2 + \left( \frac{8}{5} \right)^2 = \frac{16}{5}$$
$$f\left( \frac{9}{5}, -\frac{8}{5} \right) = \left( \frac{9}{5} - 1 \right)^2 + \left( -\frac{8}{5} \right)^2 = \frac{16}{5}$$

From the values above we observe that $f$ attains an absolute maximum of 16 and an absolute minimum of $\frac{16}{5}$. 
6. Compute the integral
\[ \int_R xy \, dx \, dy \]
over the quarter circle \( R = \{(x, y) : 0 \leq x, \ 0 \leq y, \ x^2 + y^2 \leq 1\} \). [You may use polar or Cartesian coordinates.]

Solution:

From the figure we see that the region \( D \) is bounded on the left by \( x = 0 \) and on the right by \( x = \sqrt{1 - y^2} \). The projection of \( D \) onto the \( y \)-axis is the interval \( 0 \leq y \leq 1 \). Using the order of integration \( dx \, dy \) we have:
\[ \int \int_R xy \, dx \, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy \]
\[ = \int_0^1 \left[ \frac{1}{2} x^2 y \right]_{y=0}^{y=\sqrt{1-y^2}} \, dy \]
\[ = \int_0^1 \frac{1}{2} \left( \sqrt{1-y^2} \right)^2 y \, dy \]
\[ = \frac{1}{2} \int_0^1 (1-y^2) y \, dy \]
\[ = \frac{1}{2} \int_0^1 (y - y^3) \, dy \]
\[ = \frac{1}{2} \left[ \frac{1}{2} y^2 - \frac{1}{4} y^4 \right]_0^1 \]
\[ = \frac{1}{2} \left[ \frac{1}{2} (1)^2 - \frac{1}{4} (1)^4 \right] \]
\[ = \frac{1}{8} \]
7. Compute the integral
\[
\iiint_{R} 1 \, dx \, dy \, dz
\]
over the tetrahedron
\[
R = \{(x, y, z) : 0 \leq x, \ 0 \leq y, \ 0 \leq z, \ x/3 + y/5 + z/7 \leq 1\}.
\]

**Solution:** The region of integration is shown below.
The volume of the tetrahedron is

\[
V = \iiint_R 1 \, dx \, dy \, dz
\]

\[
= \int_0^3 \int_0^{5 - 5x/3} \int_0^{7 - 7x/3 - 7y/5} 1 \, dz \, dy \, dx
\]

\[
= \int_0^3 \int_0^{5 - 5x/3} \left(7 - \frac{7}{3}x - \frac{7}{5}y\right) \, dy \, dx
\]

\[
= \int_0^3 \left[7y - \frac{7}{3}xy - \frac{7}{10}y^2\right]_0^{5 - 5x/3} \, dx
\]

\[
= \int_0^3 \left[7\left(5 - \frac{5}{3}x\right) - \frac{7}{3}x\left(5 - \frac{5}{3}x\right) - \frac{7}{10}\left(5 - \frac{5}{3}x\right)^2\right] \, dx
\]

\[
= \int_0^3 \left(35 - \frac{35}{3}x - \frac{35}{3}x + \frac{35}{9}x^2 - \frac{35}{2} + \frac{35}{3}x + \frac{35}{18}x^2\right) \, dx
\]

\[
= \int_0^3 \left(\frac{35}{2}x - \frac{35}{3}x^2 + \frac{35}{18}x^3\right) \, dx
\]

\[
= \left[\frac{35}{2}x^2 - \frac{35}{6}x^3 + \frac{35}{54}x^4\right]_0^3
\]

\[
= \frac{35}{2}(3) - \frac{35}{6}(3)^2 + \frac{35}{54}(3)^3
\]

\[
= \frac{35}{2}
\]
8. Find an equation for the tangent plane to the surface defined by $xy^2 + 2z^2 = 12$ at the point $(1, 2, 2)$.

**Solution:** We use the following formula for the equation for the tangent plane:

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

because the equation for the surface is given in implicit form. Note that $\mathbf{n} = \nabla f(a, b, c) = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle$ is a vector normal to the surface $f(x, y, z) = C$ and, thus, to the tangent plane at the point $(a, b, c)$ on the surface.

The partial derivatives of $f(x, y, z) = xy^2 + 2z^2$ are:

$$f_x = y^2$$

$$f_y = 2xy$$

$$f_z = 4z$$

Evaluating these derivatives at $(1, 2, 2)$ we get:

$$f_x(1, 2, 2) = 2^2 = 4$$

$$f_y(1, 2, 2) = 2(1)(2) = 4$$

$$f_z(1, 2, 2) = 4(2) = 8$$

Thus, the tangent plane equation is:

$$4(x - 1) + 4(y - 2) + 8(z - 2) = 0$$
9. Compute the integral
\[ \oint_C (3x^2 + y) \, dx + (x^2 + y^3) \, dy \]
over the counterclockwise boundary of the rectangle
\[ R = \{(x, y) : 0 \leq x \leq 3, \ 0 \leq y \leq 2\} \]
using Green’s theorem or otherwise.

**Solution:** Green’s Theorem states that
\[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA \]
where \( R \) is the region enclosed by \( C \). The integrand of the double integral is:
\[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \frac{\partial}{\partial x} (x^2 + y^3) - \frac{\partial}{\partial y} (3x^2 + y) \]
\[ = 2x - 1 \]
Thus, the value of the integral is:
\[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA \]
\[ = \iint_R (2x - 1) \, dA \]
\[ = \int_0^3 \int_0^2 (2x - 1) \, dy \, dx \]
\[ = \int_0^3 \left[ 2xy - y \right]_0^2 \, dx \]
\[ = \int_0^3 (4x - 2) \, dx \]
\[ = \left[ 2x^2 - 2x \right]_0^3 \]
\[ = 2(3)^2 - 2(3) \]
\[ = 12 \]