1. (5 points) This question concerns the matrix

\[ B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}. \]

**Solution:** For use in solving (a)-(c) below, put \( B^T \) into row reduced echelon form:

\[ R' = \text{RREF}(B^T) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Note: An alternate solution to this problem might use the RREF of \( B \) and the elimination matrix \( E \).

(a) (2 points) Find a basis for \( C(B^T) \), the row space of \( B \).

**Solution:** The pivot columns of \( B^T \) form a basis of \( C(B^T) \):

\[ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}. \]

In fact, any two columns of \( B^T \) (i.e. rows of \( B \)) form a basis of \( C(B^T) \). (For some other rank 2 matrices, one would need to choose the pair more carefully.)

(b) (2 points) Find a basis for \( N(B^T) \), the left null space of \( B \).

**Solution:** The null space of \( B^T \) is one-dimensional, and a basis is computed from the free column of \( R' \):

\[ \begin{pmatrix} -F \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}. \]

To check this, notice that the sum of the first and last rows of \( B \) is equal to twice the middle row.

(c) (1 point) Of the four fundamental subspaces associated to \( B \), which one can also be described as \( C(B^T) \perp \), the orthogonal complement of the row space?

**Solution:** The null space, \( N(B) \), is the orthogonal complement of \( C(B^T) \). The orthogonality of rows of \( B \) and vectors \( \mathbf{x} \in N(B) \) is expressed in the definition of the null space, \( B \mathbf{x} = \mathbf{0} \).
2. (6 points) Evaluate each determinant, or explain why it is not defined.
   (a) (1 point) $\det(-3)$
   
   **Solution:** 
   
   (b) (1 point) $\det \begin{pmatrix} 0 & 5 \\ 5 & 27 \end{pmatrix}$
   
   **Solution:** $-25$
   
   (c) (1 point) $\begin{vmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$
   
   **Solution:** Not defined, because the matrix is rectangular.
   
   (d) (1 point) $\begin{vmatrix} 0 & 1 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix}$
   
   **Solution:** $-1$ (One possible shortcut: swap rows 1,2 and subtract row 4 from row 5 to obtain upper triangular with all 1s on the diagonal. Hence $-\det = 1$.)
   
   (e) (1 point) $\det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$
   
   **Solution:** 1
   
   (f) (1 point) $\det A$, where $A$ is the $4 \times 4$ matrix with entries $a_{ij} = \begin{cases} 1, & \text{if } i + j = 5, \\ 0, & \text{otherwise}. \end{cases}$
   
   **Solution:**
   
   $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1$
   
   (Two row exchanges to obtain the identity.)
   
3. (5 points) The matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 1 & 1 \\ 6 & 3 & 1 & 2 \end{pmatrix}$ has determinant $|A| = -2$.
   
   (a) (1 point) Is $A$ invertible? *Explain your answer.*
   
   **Solution:** Yes, because the determinant is nonzero. A matrix is singular if and only if its determinant is zero.
(b) (1 point) Is there any \( b \in \mathbb{R}^4 \) such that \( Ax = b \) does not have a solution? If so, give an example. If not, explain why.

**Solution:** No, there is always a solution, because \( A \) is invertible. In fact, \( x = A^{-1}b \) is the unique solution.

(c) (3 points) Use Cramer’s rule to compute \( x_4 \) such that \( A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix} \).

**Solution:** Using a column operation and two cofactor expansions:

\[
|B_4| = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 0 \\ 6 & 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -3.
\]

Therefore,

\[ x_4 = \frac{|B_4|}{|A|} = -\frac{3}{2} = \frac{3}{2} \]

4. (7 points) This question concerns the matrix \( D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \) and the vector \( b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \).

(a) (3 points) Find three orthonormal vectors \( q_1, q_2, q_3 \) in \( \mathbb{R}^3 \) such that \( C(D) = \text{Span}(q_1, q_2) \).

**Solution:** Apply Gram-Schmidt to the columns of the invertible matrix

\[
D' = (a_1 \ a_2 \ a_3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}
\]

and thus obtain orthonormal vectors \( q_1, q_2, q_3 \) with the first two spanning \( C(D) \).

\[
A_1 = a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

\[
A_2 = a_2 - \frac{A_1^T a_2}{A_1^T A_1} A_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}
\]

\[
A_3 = a_3 - \frac{A_1^T a_3}{A_1^T A_1} A_1 - \frac{A_2^T a_3}{A_2^T A_2} A_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\]
\[ q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad q_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad q_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \]

(b) (2 points) There is no solution to \( Dx = b \). Find the least squares approximate solution \( \hat{x} \).

Solution:
\[
\hat{x} = (D^T D)^{-1} D^T b = \left[ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}
\]

As a check, notice that the error vector \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \) is orthogonal to both columns of \( D \), as it should be.

(c) (1 point) What is the projection of \( b \) onto \( C(D) \)?

Solution:
\[
p = D\hat{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}
\]

(d) (1 point) What is the projection of \( b \) onto \( C(D)^\perp \)?

Solution:
\[
e = b - p = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\]

5. (6 points)

(a) (1 point) Suppose \( Q \) is an orthogonal \( n \times n \) matrix. Is \( \lambda = 0 \) an eigenvalue of \( Q \)? Explain your answer.

Solution: A matrix with zero as an eigenvalue is singular (since the eigenvector is in the null space), but an orthogonal matrix is invertible (in fact, \( Q^{-1} = Q^T \)). Therefore, zero is not an eigenvalue of \( Q \).
(b) (3 points) Find the eigenvalues and eigenvectors of the matrix \[
\begin{pmatrix}
5 & 0 & 1 \\
0 & 5 & 0 \\
1 & 0 & 5
\end{pmatrix}
\].

**Solution:** This matrix is \(5I + A\), where \(A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\).

Clearly \(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\) is in the null space of \(A\), so it is an eigenvector of \(A\) with \(\lambda = 0\).

Since the matrix \(A\) switches the first and last components of a vector (while setting the middle equal to zero), it also has eigenvalues \(\lambda = 1\) and \(\lambda = -1\) with eigenvectors \(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\). These are similar to the case of the \(2 \times 2\) permutation matrix \(P_{12}\).

Adding \(5I\) to a matrix keeps the eigenvectors the same but adds 5 to each eigenvalue. Thus for the matrix in question:

\[
\begin{array}{c|ccc}
\lambda & 4 & 5 & 6 \\
\hline
x & \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\end{array}
\]

As a check, notice that the sum of the eigenvalues is 15, and the product is 120, which agree with the trace and determinant of the given matrix (respectively).

Note: One could also find these eigenvalues and eigenvectors by the usual algorithm—computing the characteristic polynomial, finding its roots, then finding bases for the eigenspaces. The solution given here is a possible shortcut, based on knowledge of the \(2 \times 2\) permutation matrix.

(c) (2 points) Let \(V\) be the subspace of \(\mathbb{R}^4\) consisting of all vectors orthogonal to \(\begin{pmatrix} 15 \sqrt{2} \\ -520 \\ 2008 \sqrt{17} \\ -15 \sqrt{34} \end{pmatrix}\).

Let \(P\) be the \(4 \times 4\) projection matrix onto \(V\). Compute \(\text{tr}(P)\) and \(\det(P)\), and explain your reasoning.

**Solution:** Note that \(\text{dim}(V) = 3\). Since \(P\) is a projection onto a 3-dimensional subspace of \(\mathbb{R}^4\), its eigenvalues are 0, 1, 1, 1 (listed with multiplicity). Therefore

\[
\text{tr}(P) = 0 + 1 + 1 + 1 = 3
\]

\[
\det(P) = 0 \cdot 1 \cdot 1 \cdot 1 = 0.
\]

Of course it would be impractical to calculate these (by hand) from the actual matrix

\[
P = \frac{1}{4302931}
\begin{pmatrix}
4302481 & 7800 \sqrt{2} & -30120 \sqrt{2} & -15 \sqrt{34} \\
7800 \sqrt{2} & 4032531 & 1044160 & 520 \sqrt{17} \\
-30120 \sqrt{2} & 1044160 & 270867 & -2008 \sqrt{17} \\
-15 \sqrt{34} & 520 \sqrt{17} & -2008 \sqrt{17} & 4302914
\end{pmatrix}
\].