If $C = \{0\}$, then
$$Q(C) = \sum_{n=0}^{\infty} p^n (1 - p)^{1-n} = 1 - p.$$

If $C = \{x : 1 \leq x \leq 2\}$, then $Q(C) = f(1) = p$. 

**Example 1.2.24.** Let $C$ be a one-dimensional set and let
$$Q(C) = \int_{C} e^{-x}dx.$$
Thus, if $C = \{x : 0 \leq x < \infty\}$, then
$$Q(C) = \int_{0}^{\infty} e^{-x}dx = 1.$$

If $C = \{x : 1 \leq x \leq 2\}$, then
$$Q(C) = \int_{1}^{2} e^{-x}dx = e^{-1} - e^{-2};$$
if $C_1 = \{x : 0 \leq x \leq 1\}$ and $C_2 = \{x : 1 < x \leq 3\}$, then
$$Q(C_1 \cup C_2) = \int_{0}^{1} e^{-x}dx + \int_{1}^{3} e^{-x}dx = Q(C_1) + Q(C_2).$$

If $C = C_1 \cup C_2$, where $C_1 = \{x : 0 \leq x \leq 2\}$ and $C_2 = \{x : 1 \leq x \leq 3\}$, then
$$Q(C) = Q(C_1 \cup C_2) = \int_{0}^{3} e^{-x}dx = \int_{0}^{1} e^{-x}dx + \int_{1}^{3} e^{-x}dx$$
$$= Q(C_1) + Q(C_2) - Q(C_1 \cap C_2).$$

**Example 1.2.25.** Let $C$ be a set in $n$-dimensional space and let
$$Q(C) = \int_{C} dx_1 dx_2 \cdots dx_n.$$
If $C = \{(x_1, x_2, \ldots, x_n) : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}$, then
$$Q(C) = \int_{0}^{1} \int_{0}^{x_2} \cdots \int_{0}^{x_n} dx_1 dx_2 \cdots dx_n$$
$$= \frac{1}{n!},$$
where $n! = n(n - 1) \cdots 3 \cdot 2 \cdot 1$.

EXERCISES

1.2.1. Find the union $C_1 \cup C_2$ and the intersection $C_1 \cap C_2$ of the two sets $C_1$ and $C_2$, where:

(a) $C_1 = \{0, 1, 2\}$, $C_2 = \{2, 3, 4\}$.
(b) $C_1 = \{x : 0 < x < 2\}$, $C_2 = \{x : 1 \leq x < 3\}$.
(c) $C_1 = \{(x, y) : 0 < x < 2, 1 < y < 2\}$, $C_2 = \{(x, y) : 1 < x < 3, 1 < y < 3\}$.

1.2.2. Find the complement $C^c$ of the set $C$ with respect to the space $C$ if:

(a) $C = \{x : 0 < x < 1\}$, $C^c = \{x : \frac{1}{2} < x < 1\}$.
(b) $C = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$, $C = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.
(c) $C = \{(x, y) : |x| + |y| \leq 2\}$, $C = \{(x, y) : x^2 + y^2 < 2\}$.

1.2.3. List all possible arrangements of the four letters $m, a, r$, and $y$. Let $C_1$ be the collection of the arrangements in which $y$ is in the last position. Let $C_2$ be the collection of the arrangements in which $m$ is in the first position. Find the union and the intersection of $C_1$ and $C_2$.

1.2.4. Referring to Example 1.2.18, verify De Morgan's Laws (1.2.1) and (1.2.2) by using Venn diagrams and then prove that the laws are true. Generalize the laws to arbitrary unions and intersections.

1.2.5. By the use of Venn diagrams, in which the space $C$ is the set of points enclosed by a rectangle containing the circles, compare the following sets. These laws are called the distributive laws.

(a) $C_1 \cap (C_2 \cup C_3)$ and $(C_1 \cap C_2) \cup (C_1 \cap C_3)$.
(b) $C_1 \cup (C_2 \cap C_3)$ and $(C_1 \cup C_2) \cap (C_1 \cup C_3)$.

1.2.6. If a sequence of sets $C_1, C_2, C_3, \ldots$ is such that $C_k \subset C_{k+1}$, $k = 1, 2, 3, \ldots$, the sequence is said to be a nondecreasing sequence. Give an example of this kind of sequence of sets.

1.2.7. If a sequence of sets $C_1, C_2, C_3, \ldots$ is such that $C_k \supset C_{k+1}$, $k = 1, 2, 3, \ldots$, the sequence is said to be a nonincreasing sequence. Give an example of this kind of sequence of sets.

1.2.8. If $C_1, C_2, C_3, \ldots$ are sets such that $C_k \subset C_{k+1}$, $k = 1, 2, 3, \ldots$, then $\lim_{k \to \infty} C_k$ is defined as the union $C_1 \cup C_2 \cup C_3 \cup \ldots$. Find $\lim_{k \to \infty} C_k$ if:

(a) $C_k = \{x : 1/k \leq x \leq 3 - 1/k\}$, $k = 1, 2, 3, \ldots$.
(b) $C_k = \{(x, y) : 1/k \leq x^2 + y^2 \leq 4 - 1/k\}$, $k = 1, 2, 3, \ldots$. 


1.2.9. If $C_1, C_2, C_3, \ldots$ are sets such that $C_k \supset C_{k+1}$, $k = 1, 2, 3, \ldots$, then $\lim_{k \to \infty} C_k$ is defined as the intersection $C_1 \cap C_2 \cap C_3 \cap \ldots$. Find $\lim_{k \to \infty} C_k$ if:

(a) $C_k = \{x : 2 - 1/k < x \leq 2\}$, $k = 1, 2, 3, \ldots$.

(b) $C_k = \{x : 2 < x \leq 2 + 1/k\}$, $k = 1, 2, 3, \ldots$.

(c) $C_k = \{(x, y) : 0 \leq x^2 + y^2 \leq 1/k\}$, $k = 1, 2, 3, \ldots$.

1.2.10. For every one-dimensional set $C$, define the function $Q(C) = \sum q(x)$, where $q(x) = \{q, q\}$, $x = 0, 1, 2, \ldots$, zero elsewhere. If $C_1 = \{x : x = 0, 1, 2, 3\}$ and $C_2 = \{x : x = 0, 1, 2, \ldots\}$, find $Q(C_1)$ and $Q(C_2)$. Hint: Recall that $S_n = a + ar + \cdots + ar^{n-1} = a(1 - r^n)/(1 - r)$ and, hence, it follows that $\lim_{n \to \infty} S_n = a/(1 - r)$ provided that $|r| < 1$.

1.2.11. For every one-dimensional set $C$ for which the integral exists, let $Q(C) = \int_C f(x) \, dx$, where $f(x) = 6x(1 - x)$, $0 < x < 1$; zero elsewhere; otherwise, let $Q(C)$ be undefined. If $C_1 = \{x : \frac{1}{2} < x < \frac{3}{2}\}$, $C_2 = \{\frac{1}{2}\}$, and $C_3 = \{x : 0 < x < 10\}$, find $Q(C_1), Q(C_2)$, and $Q(C_3)$.

1.2.12. For every two-dimensional set $C$ contained in $R^2$ for which the integral exists, let $Q(C) = \int_C f(x^2 + y^2) \, dx \, dy$. If $C_1 = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$, $C_2 = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$, and $C_3 = \{(x, y) : x^2 + y^2 \leq 1\}$, find $Q(C_1), Q(C_2)$, and $Q(C_3)$.

1.2.13. Let $C$ denote the set of points that are interior to or on the boundary of a square with opposite vertices at the points $(0, 0)$ and $(1, 1)$. Let $Q(C) = \int_C f(x, y) \, dx \, dy$.

(a) If $C = C_1$, the set is $\{(x, y) : 0 < x < y < 1\}$, compute $Q(C)$.

(b) If $C = C_2$, the set is $\{(x, y) : 0 < x < y < 1\}$, compute $Q(C)$.

(c) If $C = C_3$, the set is $\{(x, y) : 0 < x < y < 1\}$, compute $Q(C)$.

1.2.14. Let $C$ be the set of points interior to or on the boundary of a cube with edge of length 1. Moreover, say that the cube is in the first octant with one vertex at the point $(0, 0, 0)$ and an opposite vertex at the point $(1, 1, 1)$. Let $Q(C) = \int_C f(x, y, z) \, dx \, dy \, dz$.

(a) If $C = C_1$, the set is $\{(x, y, z) : 0 < x < y < z < 1\}$, compute $Q(C)$.

(b) If $C = C_2$, the set is $\{(x, y, z) : 0 < x < y < z < 1\}$, compute $Q(C)$.

1.2.15. Let $C$ be the subset $\{(x, y, z) : 0 < x = y < z < 1\}$, compute $Q(C)$.

1.3. The Probability Set Function

1.3.1. Let $C$ denote the sample space. What should be our collection of events? As discussed in Section 2, we are interested in assigning probabilities to events, complements of events, and union and intersection of events (i.e., compound events). Hence, we want our collection of events to include these combinations of events. Such a collection of events is called a $\sigma$-field of subsets of $C$, which is defined as follows.

Definition 1.3.1 ($\sigma$-Field). Let $B$ be a collection of subsets of $C$. We say $B$ is a $\sigma$-field if

1. $\phi \in B$, (B is not empty).
2. If $C \in B$ then $C^c \in B$, (B is closed under complements).
3. If the sequence of sets $\{C_1, C_2, \ldots\}$ is in $B$ then $\bigcup_{n=1}^{\infty} C_n \in B$, (B is closed under countable unions).

Note by (1) and (2), a $\sigma$-field always contains $\phi$ and $C$. By (2) and (3), it follows from DeMorgan's laws that a $\sigma$-field is closed under countable intersections, besides countable unions. This is what we need for our collection of events. To avoid confusion please note the equivalence: $C \subset C$. Then

the statement $C$ is an event is equivalent to the statement $C \in B$.

We will use these expressions interchangeably in the text. Next, we present some examples of $\sigma$-fields.

1. Let $C$ be any set and let $C \subset C$. Then $B = \{C, C^c, \phi, C\}$ is a $\sigma$-field.

2. Let $C$ be any set and let $B$ be the power set of $C$, (the collection of all subsets of $C$). Then $B$ is a $\sigma$-field.

3. Suppose $D$ is a nonempty collection of subsets of $C$. Consider the collection of events,

$$B = \cap_{E} : D \subset E \text{ and } E \text{ is a } \sigma\text{-field}.$$ (1.3.1)

As Exercise 1.3.20 shows, $B$ is a $\sigma$-field. It is the smallest $\sigma$-field which contains $D$; hence, it is sometimes referred to as the $\sigma$-field generated by $D$.

4. Let $C = R$, where $R$ is the set of all real numbers. Let $Z$ be the set of all open intervals.