1 HOMEWORK ASSIGNMENT 1
Assigned 1-9-15 – Due 1-21-15

Do the following problems from [1]: §1.1.1 p’4 : 1, 2; §1.2.3, p’7 : 1–5, §1.3.6 p’11–12 : 1,2,3.

2 HOMEWORK ASSIGNMENT 2
Assigned 1-21-15 – Due 1-28-15

a. 3 problems from §1.4.1 from [1]
b. 4 problems from §1.4.2 from [1].
c. Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$. The Vandermonde matrix is given as

$$V(z_1, \ldots, z_n) := \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{n-1} \end{bmatrix} \in \mathbb{C}^{n \times n}.$$

Show that $\det V(z_1, \ldots, z_n)$, called the Vandermonde determinant is equal to $\prod_{1 \leq i < j \leq n} (z_j - z_i)$.

d. Let $\sigma \in S_5$ be defined as $\sigma(1) = 3, \sigma(2) = 5, \sigma(3) = 1, \sigma(4) = 4, \sigma(5) = 2$. Find $\text{sign}(\sigma)$.

3 HOMEWORK ASSIGNMENT 3
Assigned 1-25-15 – Due 2-4-15

I. $A, B \in \mathbb{F}^{n \times m}$ are called congruent if $A = TBT^\top$ for some $T \in \text{GL}(n, \mathbb{F})$. Show

1. Congruence in $\mathbb{F}^{n \times n}$ is an equivalence relation.
2. Show that any two congruent matrices have the same rank

II. Problems 2, 4 - 7 on page 21 [1].

4 HOMEWORK ASSIGNMENT 4
Assigned 2-4-15 – Due 2-11-15


5 HOMEWORK ASSIGNMENT 5
Assigned 2-11 – Due 2-18-15

Problems 1(a,c), 2, page 33. (A is called noderogatory if the minimal polynomial of A equal to the characteristic polynomial of A.)

Additional problems:

1. Let $A \in \mathbb{F}^{n \times n}, B \in \mathbb{F}^{m \times m}$. Assume that $f, g \in \mathbb{F}[t]$ are the minimal polynomials of $A, B$ respectively. Form $C = \text{diag}(A, B) := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Let $h$ be gcd, the greatest common divisor, of $f$ and $g$, which is assumed to be monic. Show that $fg^h$ is the minimal polynomial of $C$.

2. Find the characteristic and the minimal polynomials of the following matrices

$$
\begin{bmatrix}
2 & 2 & -5 \\
3 & 7 & -15 \\
1 & 2 & -4
\end{bmatrix},
\begin{bmatrix}
2 & 5 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 4 & 2 & 0 \\
0 & 0 & 3 & 5 & 0 \\
0 & 0 & 0 & 0 & 7
\end{bmatrix}.
$$

3. Show that two similar matrices have the same minimal polynomial.

4. Show that $
\begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
$ have different characteristic polynomials, but the same minimal polynomial.

5. Show that the square matrices $A$ and $A^\top$ have the same minimal polynomial.

6. Let $A \in \mathbb{F}^{n \times n}$ and assume that $f(t) \in \mathbb{F}[t]$ is an irreducible monic polynomial for which $f(A) = 0$. Show that $f$ is the minimal polynomial of $A$.

6 HOMEWORK ASSIGNMENT 6
Assigned 2-18-15 – Due 2-25-15

Problems 1–3, 4b, §2.5, page 39. (Weyr characteristic is defined in Definition 2.30 on p’37 of [1]).

Problem 1. Suppose that the characteristic and the minimal polynomial of a linear operator $T$ are as below. Find all possible Jordan canonical forms of $T$. 

2
1. \( f(t) = (t - 2)^4(t - 5)^3, g(t) = (t - 2)^4(t - 5)^3, \)
2. \( f(t) = (t - 2)^4(t - 5)^3, g(t) = (t - 2)^2(t - 5)^3, \)
3. \( f(t) = (t - 2)^4(t - 5)^3, g(t) = (t - 2)(t - 5). \)

Problem 2. Find all possible Jordan forms for all \( 8 \times 8 \) matrices having \( x^2(x - 1)^3 \) as a minimal polynomial.

Problem 3.
   a. Show that if the characteristic polynomial of \( A \in \mathbb{F}^{n \times n} \) splits to linear factors in \( \mathbb{F} \), i.e. \( \det(zI - A) = \prod_{j=1}^{n} (z - \lambda_j) \), then \( A \) is similar to \( A^\top \). (Hint: Let \( \mathbb{F}_1 \) be a finite extension of \( \mathbb{F} \), where \( \det(zI - A) \) splits to linear factors. Then by part a, show that \( A \) and \( A^\top \) are similar over \( \mathbb{F}_1 \). So there exists a matrix \( X \in \mathbb{F}_1^{n \times n} \) such that \( AX -XA^\top = 0 \) and \( \det X \neq 0 \). Deduce now that one can choose \( X \) in \( \mathbb{F}^{n \times n} \) such that \( \det X \neq 0 \).)

Problem 4. Recall that a matrix \( A \in \mathbb{F}^{n \times n} \) is called diagonable if \( A \) is similar to a diagonal matrix over \( \mathbb{F} \). A linear operator \( T : \mathbb{V} \rightarrow \mathbb{V} \) is called diagonable if there is a basis in \( \mathbb{V} \) such that \( T \) is represented by a diagonal matrix. Show

1. \( A \) is diagonable over \( \mathbb{F} \) if and only if \( \det(zI - A) \) splits to linear factors over \( \mathbb{F} \), and the minimal polynomial of \( A \) has simple roots.

2. \( A \) is diagonable if the roots of \( \det(zI - A) \) are in \( \mathbb{F} \) whenever \( (T - \lambda I)^m v = 0 \), for some positive integer \( m \), then \( (T - \lambda I) v = 0 \).

3. Suppose that the linear operator \( T \) is a projection, i.e. \( T^2 = T \). Then \( T \) is diagonable.

4. Assume that \( T, Q \in \mathbb{F}^{n \times n} \) are projections. Then \( T \) and \( Q \) are similar if and only if rank \( T = \text{rank} \ Q \).

5. Let \( n > 1 \) be an integer, and consider the matrices \( A = 11^\top \in \mathbb{F}^{n \times n}, 1 = (1, \ldots, 1)^\top \in \mathbb{F}^n \) and the diagonal matrix \( \text{diag}(n, 0, \ldots, 0) \in \mathbb{F}^{n \times n} \). Then \( A \) and \( B \) are similar if and only if the characteristics of \( \mathbb{F} \) does not divide \( n \).

7 HOMEWORK ASSIGNMENT 7
Assigned 2-24-15 – Due 3-4-15

1. Problems 1 - 3 in the end of §2.6.

2. Problems 1 - 2 in the end of §3.1.

8 HOMEWORK ASSIGNMENT 8
Assigned 3-4-15 – Due 3-13-15

Note: (*) are more advanced problems (can be skipped for the test.)
1. For the two matrices given in Problem 2 of HW 5 and the $4 \times 4$ matrix $C$ given in Problem 3 of §2.3, (page 34 in the current version on the web) find:

(a) $A^l$ for $l \in \mathbb{N}$;
(b) $e^{At}$.

2. Recall that a stochastic matrix $A \in \mathbb{R}^{n \times n}$ has nonnegative entries such that the sum of the entries in each row is 1. Assume that $A$ is stochastic.

(a) Show that $A^l$ is stochastic for each $l \in \mathbb{N}$.
(b) Show that each eigenvalue $\lambda$ of $A$ satisfies $|\lambda| \leq 1$. Furthermore that if $|\lambda| = 1$ then $\lambda$ is a simple root of the minimal polynomial.
(c) (*) Assume that $\lambda$ is an eigenvalue of $A$ and $|\lambda| = 1$. Is it always true that $\lambda$ is a simple root of the characteristic polynomial of $A$. (Either prove, or give a counterexample.)
(d) Show that 1 is an eigenvalue of $A$. Find one corresponding eigenvector.
(e) What is the necessary and sufficient condition for $\lim_{l \to \infty} A^l = B$.
(f) Give an example of stochastic matrix such that the sequence $A^l, l \in \mathbb{N}$ does not converge to any limit.
(g) (*) Show that for any stochastic $A$ the Cesaro sequence $\frac{1}{l} \sum_{i=0}^{l-1} A^i$ converges to $C$. Furthermore if $\lim_{l \to \infty} A^l = B$ then $C = B$.

9 HOMEWORK ASSIGNMENT 9
Assigned 3-24-15 – Due 4-1-15

Problems 1 to 6 are from Math 320 course. Consult with Chapter 4 in my Math 320 notes:
http://homepages.math.uic.edu/~friedlan/math320lecS13t.pdf

Write down briefly the steps or formulas needed to solve Problems 1-6 but do not do all the computations.

1. Let $u = (1, -1, 1, -1)^T$, $v = (2, 0, -2, 1)^T$. Find

(a) The cosine of the angle between $u$ and $v$.
(b) The scalar and the vector projection of $v$ on $u$.
(c) A basis to the orthogonal complement of $U := \text{span}(u, v)$.
(d) The projection of the vector $(1, 1, 0, 0)^T$ on $U$ and $U^\perp$.

2. Let $A \in \mathbb{R}^{4 \times 3}$. Assume that the vector $(1, -1, 1, -1)^T$ is a vector in the column space of $A$. Is it possible that a vector $(2, 0, -2, 1)^T$ is in the null space of $A^T$? If yes give an example of such a matrix. If not, justify why.

3. Consider the overdetermined system

$$
\begin{align*}
    x_1 + x_2 + x_3 &= 4 \\
    -x_1 + x_2 + x_3 &= 0 \\
    -x_2 + x_3 &= 1 \\
    x_1 + x_3 &= 2
\end{align*}
$$
(a) Is this system solvable?
(b) Find the least squares solution of this system.
(c) Find the projection of \((4,0,1,2)^\top\) on the column space of the coefficient
matrix \(A \in \mathbb{R}^{4 \times 3}\) of this system.

Let \((-1,0), (0,1), (1,3), (2,9)\) be four points in the plane \((x,y)\) Find

4. (a) The best least squares fit by a linear function \(y = ax + b\).
(b) The best least squares by a quadratic polynomial \(y = ax^2 + bx + c\).
(c) Explain briefly why there exist a unique cubic polynomial \(y = ax^3+bx^2+cx+d\) passing through these four points.

5. Let \((-1,0), (0,1), (1,3), (2,9)\) be four points in the plane \((x,y)\) Find

4. (a) The best least squares fit by a linear function \(y = ax + b\).
(b) The best least squares by a quadratic polynomial \(y = ax^2 + bx + c\).
(c) Explain briefly why there exist a unique cubic polynomial \(y = ax^3+bx^2+cx+d\) passing through these four points.

5. Let \(a \leq t_1 < t_2 < \ldots < t_n \leq b\) be \(n\) points in the interval \([a,b]\). For any two
continuous functions \(f, g \in C[a,b]\) define \(\langle f, g \rangle := \sum_{i=1}^n f(t_i)g(t_i)\). Let \(P_m\) be
the vector space of all polynomials of degree at most \(m-1\).

(a) Show that for \(m \leq n \langle , \rangle\) is an inner product on \(P_m\).
(b) Is \(\langle , \rangle\) an inner product on \(P_{n+1}\)? Justify!

6. For the inner product \(\langle f, g \rangle := \int_{-1}^1 f(x)g(x)dx\) on \(C[-1,1]\) Find the cosine of
the angle between \(f(x) = 1\) and \(g(x) = e^x\).

7. PROBLEMS 2, 6 on p’ 61 - 62 in the end of §4.5

8. Problems 1, 2 and 3 on page 64.

10 HOMEWORK ASSIGNMENT 10
Assigned 4-3-15 – Due 4-8-15

1. page 69: Problems: 9(a,b,c), (special orthogonal means determinant one), 10a, 12.

2. Problem 3 and 4 on pages 75 - 76.

3. Assume that \(A\) a real symmetric matrix. Denote by \(\iota_+(A)\) be the number
of positive eigenvalues, \(\iota_0(A)\) the number of zero eigenvalues, \(\iota_-(A)\) be the
number of negative eigenvalues. Denote \(\iota(A) := (\iota_+(A), \iota_0(A), \iota_-(A))\). Show.

(a) Show that \(i_+(A)\) is the dimension of any subspace \(U \subset \mathbb{R}^n\) such that
\(x^\top Ax > 0\) for each nonzero \(x\) in \(U\). (Hint: Use the convoy principle.)
(b) Show that \(i_-(A)\) is the dimension of any subspace \(U \subset \mathbb{R}^n\) such that
\(x^\top Ax < 0\) for each nonzero \(x\) in \(U\).
(c) rank \(A = \iota_+(A) + \iota_-(A)\).
(d) A symmetric \(B \in \mathbb{R}^{n \times n}\) is called congruent to \(A\) if \(B = QAQ^\top\) for some
invertible matrix \(Q\). Show that two symmetric matrices are congruent
if and only if \(\iota(A) = \iota(B)\). (This result is called the Sylvester law of
inertia. This is the content of Problems 6 and 7 on page 76 for the
hermitian case.)
4. Let $A = [a_{pq}] \in \mathbb{C}^{n \times n}$ be a hermitian matrix. Rearrange the diagonal entries of $A$, $a_{11}, a_{22}, \ldots, a_{nn}$ in a nonincreasing way: $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. Show 

(a) $\lambda_1(A) \geq \alpha_1, \lambda_n(A) \leq \alpha_n$. **Hint:** Use the maximum and minimum characterization of $\lambda_1(A), \lambda_n(A)$.

(b) Show that $\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{n} \lambda_i(A)$ for $k = 1, \ldots, n$. What happens for $k = n$? **Hint:** Use the convoy principle.

(c) Show that $|\lambda_j(A)| \leq \sqrt{\sum_{p=q=1}^{n} |a_{pq}|^2}$ for each $j = 1, \ldots, n$. For which kind of matrices and for which $j$ we have equality in this inequality?

11 HOMEWORK ASSIGNMENT 11

**Assigned 4-11 – Due 4-17-15**

A.

1. Let $A, B \in \mathbb{H}_n$. Assume that $B > 0$. Show that $AB$ is diagonalable and has real eigenvalues. Which part of this statement remains correct if (a) $B \geq 0$, (b) $B$ is just hermitian.

2. Let $A \in \mathbb{C}^{n \times n}$ and view $A$ as a linear transformation $T$ given by $x \mapsto Ax$, from $\mathbb{C}^n$ to itself. $A$ is called symmetrizable if there exists $B > 0$ such that $T$ is selfadjoint with respect to the inner product on $\mathbb{C}^n \langle x, y \rangle := y^* B x$. Show that $A$ is symmetrizable if and only if $A$ is similar to a real diagonal matrix.

B. Problems 3,4,9 [1, p’ 86-87]. (In 9 you can assume that $A$ is a normal matrix.)

C.

1. Let $A \in \mathbb{C}^{m \times n}$. Show that $A, \bar{A}, A^*, A^\top$ have the same nonzero singular values. How their singular value decomposition related?

2. Find the singular value decomposition of the following matrices:

   \[
   (a) \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}; \quad (b) \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}.
   \]

3. Find a best rank one and rank two approximation for the two matrices in 2.

4. Assume that $A$ is a hermitian matrix. Show that the singular values of $A$ are the absolute values of the eigenvalues of $A$.

5. Assume that $A \in \mathbb{C}^{n \times n}$. Show that $AA^*$ and $A^*A$ are similar.
References

[1] S. Friedland, Linear Algebra II, Lectures Notes, Spring 2015, 
http://www2.math.uic.edu/~friedlan/lectnotesM425S15.pdf


2002.