1 Change of basis

Assume that $V$ is an $n$-dimensional vector space. Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis in $V$, as in the book [1].

**Notation:** we denote $\mathcal{B}$ by $[b_1 \; b_2 \; \ldots \; b_n]$. Then any vector $x \in V$ can be uniquely presented as $x = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$. Lay’s notation: $x_\mathcal{B} := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$.

Then we write

$$x = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = v_1 a_1 + \ldots + v_n a_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$ (1.1)

$$[b_1 \; b_2 \; \ldots \; b_n] x_\mathcal{B} = [b_1 \; b_2 \; \ldots \; b_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$ (1.2)

We now introduce another basis $C := \{c_1, \ldots, c_n\}$ in $V$.

**The transition matrix from the basis $\mathcal{B}$ to the basis $C$,** denoted in Lay as $P_{C \leftarrow \mathcal{B}}$ is formally given by the identity:

$$[b_1 \; b_2 \; \ldots \; b_n] = [c_1 \; c_2 \; \ldots \; c_n] P_{C \leftarrow \mathcal{B}}.$$ (1.2)

In words, the column $j$ of the matrix $P_{C \leftarrow \mathcal{B}}$ is $[b_j]_C$, the coordinate vector of $b_j$ with in the basis $C$. That is,

$$P_{C \leftarrow \mathcal{B}} = [[b_1]_C \; [b_2]_C \; \ldots \; [b_n]_C].$$ (1.3)

**Example 1, [1, p’ 239]:**

$$b_1 = 4c_1 + c_2, \quad b_2 = -6c_1 + c_2.$$ Then

$$P_{C \leftarrow \mathcal{B}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}.$$ (1.4)

**Theorem 1.1** Let $V$ be an $n$ dimensional vector space with bases $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ and $C := \{c_1, \ldots, c_n\}$.

1. Assume that the transition matrix from the basis $\mathcal{B}$ to $C$ is given by the matrix $P_{C \leftarrow \mathcal{B}}$ that satisfies (1.2). Then the transition matrix from the basis $C$ to $\mathcal{B}$ is given by the inverse of the matrix $P_{C \leftarrow \mathcal{B}}$

$$P_{\mathcal{B} \leftarrow C} = P_{C \leftarrow \mathcal{B}}^{-1}.$$ (1.4)
2. Assume that the coordinates of a vector \( x \) are \([x]_B\) and \([x]_C\) in the bases \( B \) and \( C \) respectively. Then

\[
[x]_C = P_{C\leftarrow B}[x]_B, \quad [x]_B = P_{C\leftarrow B}^{-1}[x]_C = P_{B\leftarrow C}[x]_C. \quad (1.5)
\]

**Proof.** 1: Multiply both sides of \((1.2)\) by \( P_{C\leftarrow B}^{-1} \) form the right to deduce that

\[
[c_1 \; c_2 \cdots c_n] = [b_1 \; b_2 \cdots b_n]P_{C\leftarrow B}^{-1} = [b_1 \; b_2 \cdots b_n]P_{B\leftarrow C}.
\]

2: Multiply both sides of \((1.2)\) by \([x]_B\) and use \((1.1)\) to obtain:

\[
x = [b_1 \; b_2 \cdots b_n][x]_B = [c_1 \; c_2 \cdots c_n](P_{C\leftarrow B}[x]_B) = [c_1 \; c_2 \cdots c_n][x]_C.
\]

This shows the first equality in \((1.5)\). Use the equality \((1.4)\) to deduce the second equality of \((1.5)\). \(\square\)

Hence in the above Example 1:

\[
P_{B\leftarrow C} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1 & 0.6 \\ -0.1 & 0.4 \end{bmatrix}.
\]

Furhtermore, suppose as in Example 1 \( x = 3b_1 + b_2 \). So \([x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\). Then \((1.5)\) yields:

\[
[x]_C = P_{C\leftarrow B}[x]_B = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.
\]

**Theorem 1.2** Let \( V \) be an \( n \) dimensional vector space with three bases \( A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}, C := \{c_1, \ldots, c_n\} \). Assume that \( P_{A\leftarrow B} \) and \( P_{A\leftarrow C} \) are the transition matrix from the basis \( B \) to \( A \) and \( C \) to \( A \) respectively. Then the transition matrix from \( B \) to \( C \) is given be the formula

\[
P_{C\leftarrow B} = P_{A\leftarrow C}^{-1}P_{A\leftarrow B}. \quad (1.6)
\]

Moreover, to compute \( P_{A\leftarrow C}^{-1}P_{A\leftarrow B} \) consider the following \( n \times (2n) \) matrix \( F := [P_{A\leftarrow C} \; P_{A\leftarrow B}] \). Perform ERO (elementry row operations) to bring \( F \) to its RREF, which is \([I_n \; G]\). Then \( G = P_{A\leftarrow C}^{-1}P_{A\leftarrow B} \).

**Proof.** Observe that

\[
[b_1, \ldots, b_n] = [a_1, \ldots, a_n]P_{A\leftarrow B}, \quad [c_1, \ldots, c_n] = [a_1, \ldots, a_n]P_{A\leftarrow C}.
\]

Use the second equality to deduce that \([a_1, \ldots, a_n] = [c_1, \ldots, c_n]P_{A\leftarrow C}^{-1} \). Subsitute this equality to the frist equality in the above equalities to obtain

\[
[b_1, \ldots, b_n] = [a_1, \ldots, a_n]P_{A\leftarrow B} = [a_1, \ldots, a_n](P_{A\leftarrow C}^{-1}P_{A\leftarrow B}).
\]

This proves \((1.6)\).

Consider the matrix \( F := [P_{A\leftarrow C} \; P_{A\leftarrow B}] \). Since \( P_{A\leftarrow C} \) the RREF of \( F \) is \([I_n \; G]\). It is obtaines by considering the product

\[
P_{A\leftarrow C}^{-1}F = [P_{A\leftarrow C}^{-1}P_{A\leftarrow C} \; P_{A\leftarrow C}^{-1}P_{A\leftarrow B}] = [I_n \; G].
\]
Hence $G = P_{A→C}^{-1}P_{A→B}$. □

Consider Example 2 [1, p'241]:

\[ b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}. \]

Observe that implicitly $A := \{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$ is the standard basis in $\mathbb{R}^2$.

Thus

\[ P_{A→B} = \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}, \quad P_{A→C} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}. \]

Hence

\[ F = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix} \]

Thus $P_{C→B} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$.

1.1 Another example

Let $B = \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \}, C = \{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \}$

Find the transition matrix from the basis $C$ to basis $B$.

**Solution:** Introduce the standard basis $A = \{ e_1, e_2 \}$ in $\mathbb{R}^2$. So $[b_1 \ b_2] = [e_1 \ e_2] [c_1 \ c_2] = [e_1 \ e_2] [3 \ 4 \ 4 \ 5]$. Hence the transition matrix is $[1 \ 1 \ 2 \ 3]^{-1} [3 \ 4 \ 4 \ 5]$.

To find this matrix get the RREF of $[1 \ 1 \ | \ 3 \ 4 \ 4 \ 5]$ which is $[1 \ 0 \ | \ 5 \ 7 \ \ -2 \ -3]$. 

Answer $[5 \ 7 \ -2 \ -3]$

References