2.2

(1) $H$ is time independent, so Schrödinger’s equation solves to

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle. \tag{1}$$

$H$ can be diagonalized as $H = U\Lambda U^\dagger$, where

$$\Lambda = -\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

so

$$e^{-iHt/\hbar} = U e^{-i\Lambda t/\hbar} U^\dagger = U \begin{pmatrix} e^{it\omega/2} & 0 \\ 0 & e^{-it\omega/2} \end{pmatrix} U^\dagger = \begin{pmatrix} \cos \frac{t\omega}{2} & \sin \frac{t\omega}{2} \\ -\sin \frac{t\omega}{2} & \cos \frac{t\omega}{2} \end{pmatrix}.$$  

Therefore, (1) yields

$$|\psi(t)\rangle = \begin{pmatrix} \cos \frac{t\omega}{2} & \sin \frac{t\omega}{2} \\ -\sin \frac{t\omega}{2} & \cos \frac{t\omega}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \frac{t\omega}{2} \\ \cos \frac{t\omega}{2} \end{pmatrix}.$$  

(2) The observable $\sigma_z$ has eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ with eigenvectors $|\lambda_1\rangle = (1,0)^T, |\lambda_2\rangle = (0,1)^T$. Thus,

$$|\psi(t)\rangle = \sin \frac{t\omega}{2} |\lambda_1\rangle + \cos \frac{t\omega}{2} |\lambda_2\rangle,$$

so the probability of observing $+1$ at time $t$ is $\sin^2(t\omega/2)$.

(3) The observable $\sigma_x$ has eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ with eigenvectors $|\lambda_1\rangle = \frac{1}{\sqrt{2}} (1,1)^T, |\lambda_2\rangle = \frac{1}{\sqrt{2}} (1,-1)^T$. Thus, if we have

$$|\psi(t)\rangle = c_1 |\lambda_1\rangle + c_2 |\lambda_2\rangle,$$  \tag{2}
then $|c_1|^2$ is the probability of observing +1 at time $t$. (??) is simply a system of 2 equations in $c_1, c_2$, which can be solved trivially to yield $c_1 = (\sin \frac{t \omega}{2} + \cos \frac{t \omega}{2}) / \sqrt{2}$. Thus, the probability of observing +1 at time $t$ is

$$|c_1|^2 = \frac{1}{2} \left( \sin \frac{t \omega}{2} + \cos \frac{t \omega}{2} \right)^2 = \frac{1}{2} \left( 1 + 2 \sin \frac{t \omega}{2} \cos \frac{t \omega}{2} \right) = \frac{1}{2}(1+\sin(tw)).$$

2.4

First assume $\rho$ is pure. Then by theorem 2.1, $\rho^2 = \rho$, so

$$\text{tr} \rho^2 = \text{tr} \rho = 1.$$  

Now assume $\text{tr} \rho^2 = 1$. Let $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ be the eigenvalues of $\rho$. Note that they are all nonnegative real numbers since $\rho$ is Hermitian and positive semidefinite. Then $\lambda_1^2 \geq \ldots \geq \lambda_n^2 \geq 0$ are the eigenvalues of $\rho^2$. Thus, we have

$$\text{tr} \rho = \lambda_1 + \ldots + \lambda_n = 1 \quad (3)$$

and

$$\text{tr} \rho^2 = \lambda_1^2 + \ldots + \lambda_n^2 = 1. \quad (4)$$

Subtracting (3) from (4), we get

$$\sum_{i=1}^{n} \lambda_i(1 - \lambda_i) = 0. \quad (5)$$

Observe that since $\text{tr} \rho = 1$, we have $0 \leq \lambda_i \leq 1$ for all $i$. Thus, each term in (3) is nonnegative. Since they sum to 0, we conclude $\lambda_i \in \{0, 1\}$ for all $i$. But by (3), exactly one of the $\lambda_i$ (namely, $\lambda_1$) must be 1 and the rest 0; therefore, the eigendecomposition of $\rho$ is

$$\rho = \sum_{i=1}^{n} \lambda_i |\lambda_i\rangle\langle \lambda_i| = \lambda_1 |\lambda_1\rangle\langle \lambda_1|,$$

so $\rho$ is pure.

2.5

Clearly $\rho$ is hermitian and $\text{tr} \rho = 1$. Its eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = (1 - p)/4, \lambda_4 = (1 + 3p)/4$, which are all $\geq 0$ for any $p \in [0, 1]$, so $\rho$ is positive semidefinite. Therefore, $\rho$ is a density matrix.
Now assume \( p > 1/3 \). Consider \( \rho \) as a \( 2 \times 2 \) block matrix of \( 2 \times 2 \) blocks \( \rho_{ij} \). Then

\[
\rho^{pt} = \begin{pmatrix}
\rho_{11}^\top & \rho_{12}^\top \\
\rho_{21} & \rho_{22}
\end{pmatrix} = \begin{pmatrix}
\frac{1+p}{4} & 0 & 0 & 0 \\
0 & \frac{1-p}{4} & \frac{p}{2} & 0 \\
0 & \frac{p}{2} & \frac{-1}{4} & 0 \\
0 & 0 & 0 & \frac{1+p}{4}
\end{pmatrix}.
\]

The eigenvalues of \( \rho^{pt} \) are \( \mu_1 = \mu_2 = \mu_3 = (1 + p)/4, \mu_4 = (1 - 3p)/4 \). Since \( p > 1/3, \mu_4 < 0 \). Thus, we have

\[
N(\rho) = \frac{\sum_{i=1}^{4} |\mu_i| - 1}{2} = \frac{\mu_1 + \mu_2 + \mu_3 - \mu_4 - 1}{2} = \frac{3p - 1}{4} > \frac{3 \cdot \frac{1}{3} - 1}{4} = 0.
\]

**2.7**

The density matrix is

\[
\rho = |\psi'\rangle\langle\psi'| = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix}.
\]

Each \( \rho_{ij} \) is a \( 2 \times 2 \) block. The partial trace over \( \mathcal{H}_1 \) is thus

\[
\text{tr}_1(\rho) = \begin{pmatrix}
\text{tr}(\rho_{11}) & \text{tr}(\rho_{12}) \\
\text{tr}(\rho_{21}) & \text{tr}(\rho_{22})
\end{pmatrix} = \frac{1}{2} I_2.
\]

**2.8**

We write

\[
\rho_1 = \frac{1}{4} |\psi_1\rangle\langle\psi_1| + \frac{3}{4} |\psi_2\rangle\langle\psi_2|.
\]

We will use \( |\psi_1\rangle, |\psi_2\rangle \) as the basis for the new Hilbert space as well. According to (2.53) we get

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} |\psi_1\rangle \otimes |\psi_1\rangle + \frac{\sqrt{3}}{2} |\psi_2\rangle \otimes |\psi_2\rangle
\]

**2.9**

Unitary transformations map orthonormal vectors to orthonormal vectors. Thus, \( U|\phi_k\rangle \) are orthonormal, so \( |\Psi'\rangle \) is a purification of \( \rho_1 \).
2.10

**Observation 1.** Let $U$ be unitary and $A$ be Hermitian and positive semidefinite. Then $B = U A U^\dagger$ is Hermitian and positive semidefinite, and $\sqrt{B} = U \sqrt{A} U^\dagger$.

**Proof.** Trivially $B$ is Hermitian. $A$ and $B$ are similar, so they have the same eigenvalues. Thus, $B$ is positive semidefinite since $A$ is.

Now observe that

$$(U \sqrt{A} U^\dagger)^2 = U \sqrt{A} U^\dagger U \sqrt{A} U^\dagger = U \sqrt{A} \sqrt{A} U^\dagger = U A U^\dagger = B.$$

$\sqrt{B}$ is the unique matrix whose square is $B$, so this proves $U \sqrt{A} U^\dagger = \sqrt{B}$. $\square$

By observation ??, we have

$$\sqrt{U \rho_1 U^\dagger U \rho_2 U^\dagger} = U \sqrt{\rho_1} U^\dagger U \sqrt{\rho_2} U^\dagger = U \sqrt{\rho_1 \rho_2} \sqrt{\rho_1} U^\dagger.$$

Since $\sqrt{\rho_1 \rho_2} \sqrt{\rho_1}$ is Hermitian and positive semidefinite, again we may apply observation ?? to get

$$\sqrt{U \rho_1 U^\dagger U \rho_2 U^\dagger} = \sqrt{U \sqrt{\rho_1 \rho_2} \sqrt{\rho_1} U^\dagger} = \sqrt{\sqrt{\rho_1 \rho_2} \sqrt{\rho_1} U^\dagger}.$$

Finally, $U \sqrt{\sqrt{\rho_1 \rho_2} \sqrt{\rho_1} U^\dagger}$ is similar to $\sqrt{\sqrt{\rho_1 \rho_2} \sqrt{\rho_1}}$, so they have the same trace.

2.11

$\rho_1$ is a diagonal matrix, so $\sqrt{\rho_1} = \text{diag}\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$ and

$$\sqrt{\rho_1 \rho_2} \sqrt{\rho_1} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = B.$$

To find $\sqrt{B}$, we first diagonalize it using methods from HW 1:

$$B = U \Lambda U^\dagger$$.
where

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \]

and \( \Lambda = \frac{1}{2} \text{diag}(1, 0, 0, 0) \). Since \( \Lambda \) is diagonal, we have \( \sqrt{\Lambda} = \frac{1}{\sqrt{2}} \text{diag}(1, 0, 0, 0) = \sqrt{2} \Lambda \) and hence

\[ \sqrt{B} = U \sqrt{\Lambda} U^\dagger = U(\sqrt{2} \Lambda) U^\dagger = \sqrt{2} U \Lambda U^\dagger = \sqrt{2} B. \]

Therefore,

\[ F(\rho_1, \rho_2) = \text{tr}(\sqrt{B}) = \frac{\sqrt{2}}{4}(1 + 1) = \frac{\sqrt{2}}{2}. \]