3.3 Adjoint Matrix

\[
\text{adj } A = \begin{bmatrix}
C_{11} & -C_{1n} \\
-C_{11} & C_{nn}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{bmatrix}
\]

\[\text{Example 3 p’ 179}\]

**Theorem** For any matrix \( A \in \mathbb{R}^{n \times n} \)

\[A(\text{adj } A) = (\text{adj } A^T)A = (\det A)I_n\]

In particular, if \( \det A \neq 0 \) then

\[A^{-1} = \frac{1}{\det A} \text{adj } A\]

**Outline of proof**

\[\det A = a_{11}c_{11} + a_{12}c_{1n} \Rightarrow (A(\text{adj } A))_{11} = 1\]

Similarly \(a_{i1}c_{i1} + a_{i2}c_{in} = 0 \Rightarrow (A(\text{adj } A))_{i1} = 0\)

We now show that \(a_{11}c_{21} + a_{1n}c_{2n} = 0\)

Look at \[
\begin{vmatrix}
a_{11} & a_{12} & -a_{1n} \\
a_{21} & a_{22} & -a_{2n} \\
a_{31} & a_{32} & -a_{3n}
\end{vmatrix}
\]

This determinant is zero, as \(B\) has two same rows.

Expand this determinant by the second row to deduce

\[\det B = a_{11}(C_{21} + a_{1n}C_{2n}) = 0\]
Proof of Cramer’s rule: \( Ax = b, \det A \neq 0 \)

\[
x = A^{-1} b = \frac{1}{\det A} (\text{adj} A) b
\]

\[x_k = \frac{1}{\det A} \left[ b_1 \begin{array} C 1 \end{array} C_{1k} + b_2 \begin{array} C 1 \end{array} C_{2k} + b_3 \begin{array} C 1 \end{array} C_{3k} \right]
\]

\[
x_2 = \begin{array} C 1 \end{array} C_{12} + b_2 \begin{array} C 1 \end{array} C_{22} + b_3 \begin{array} C 1 \end{array} C_{32}
\]

Expansion of \( \begin{array} C 1 \end{array} C_{k-1} b \begin{array} C 1 \end{array} C_{k+1} - \begin{array} C 1 \end{array} C_{1k} \begin{array} C 1 \end{array} C_{k+1} \) by k-th column

Determinants as volumes

Theorem: Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \).

Then \( \det A \) is the volume of the parallelepiped spanned by \( a_1, \ldots, a_n \) in \( \mathbb{R}^n \).

Elementary row operations:

1. \( \text{Row } i \leftrightarrow \text{Row } j \)
2. Add a multiple of row \( j \) to row \( i \)

If \( \det A = 0 \), then \( A \) is linearly dependent, \( \det A = 0 \).

Otherwise, \( A \) is invertible.

Volume spanned by columns \( U \) and \( U_{11} - \text{ann} \).
THM

\[ T: \mathbb{R}^n \to \mathbb{R}^n \] represented by \( A \in \mathbb{M}_{n,n} \)

- \( S \) is a parallelepiped in \( \mathbb{R}^n \)
- \( B = [b_1, \ldots, b_n] \)

Then
\[
\text{vol}(B) = |\det A| \text{vol}(B)
\]

\[
T(B) = [Tb_1, \ldots, Tb_n] = A B
\]

\[
\text{det } T(B) = |\det A| \text{det } B = |\det A| \text{vol}(B)
\]

This explains the change of area/volume

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}
\]

Jacobian

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Example

The volume of an ellipsoid

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

\[
\frac{u}{a} \quad \frac{v}{b} \quad \frac{w}{c}
\]

So the volume of the ellipsoid is \( \frac{4}{3}\pi abc \).