1. (20 pts) Consider the system of equations $Ax = b$ where:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Compute det $A$. Is $A$ singular or nonsingular?
(b) Compute $A^{-1}$ if possible.
(c) Write the row reduced echelon form of $A$.
(d) Find all solutions to the system $Ax = b$.

2. (20 pts) Consider the following matrix $A$:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

(a) Find the nullspace of $A$.
(b) Do the columns of $A$ form a spanning set for $\mathbb{R}^2$. Clearly explain why or why not.

3. (20 pts) Do the vectors below form a basis for $\mathbb{R}^3$? If so, explain. If not, remove as many vectors as you need to form a basis and show that the resulting set of vectors form a basis for $\mathbb{R}^3$.

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$  

4. (20 pts) Consider the following mapping $L : \mathbb{R}^2 \to \mathbb{R}^3$:

$$L(x) = \begin{bmatrix} 2x_1 \\ -x_2 \\ x_1 + x_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$  

(a) Show that $L$ is a linear transformation.
(b) Find a matrix representation for $L$ using the standard basis for $\mathbb{R}^3$ and the following basis vectors for $\mathbb{R}^2$: $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

5. (20 pts) Let $Y = \text{Span} \{x_1, x_2\}$ where: $x_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Find $Y^\perp$, the orthogonal complement of $Y$.

6. (20 pts) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Find the QR decomposition of $A$.

7. (20 pts) Find a matrix $X$ and a diagonal matrix $D$ such that $A = XDX^{-1}$, where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$
8. (20 pts) Find the solution to the system of first order ODEs:
\[
\frac{dy_1}{dt} = y_1 - 4y_2, \quad y_1(0) = 3
\]
\[
\frac{dy_2}{dt} = -y_2, \quad y_2(0) = 3.
\]

9. (20 pts.) Find the least square solution of the system \( Ax = b \):
\[
\begin{align*}
x_1 + x_2 &= 3 \\
x_1 - x_2 &= 1 \\
x_1 + 3x_2 &= -1
\end{align*}
\]
Use this solution to find the projection of \( b \) on the column space of the coefficient matrix \( A \).

10. (20 pts) Let \( A \) be the following symmetric matrix
\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( A = QDQ^\top \).