1. \( Q_k \) can be viewed as two copies \( Q_{k-1} \cup Q_k \) where:

\[
x = (x_1, \ldots, x_k, 0), \quad y = (x_1, \ldots, x_{k-1}, 1)
\]

for \( x' \in E(Q_k) \).

Each \( Q_{k-1}, Q_k \) contains \( Q_p \) and \( Q_q \) as a subgraph. So take a Hamiltonian cycle and \( Q_{k-1} Q_q \) which is in \( Q_{k-1} \).

Now you can assume that one edge is \((0, 0, 0, 1)\).

Now go down to \( Q_{k-1} \) which contains a copy of \( Q_q \) look at \( Q_p \) in \( Q_q \) and now take a Hamiltonian cycle in \( Q_p \) which says starts 10 0 0 and ends at 0 0 0 now so \( Q_p \) to get the desired cycle.

2. Proof by induction on the number of vertices in \( D = (V, A) \). \( |V| = n \)

- \( n = 2 \): A dipath. \( V \)

Assume true for \( n = k \). Suppose \( n = k + 1 \)

Take a subdigraph on \( k + 1 \) vertices \( V_1, \ldots, V_k \). It is also a tournament. By induction hypothesis there exist a Hamiltonian dipath on \( k \) vertices.

Rename the vertices \( V_1, \ldots, V_k \) to \( 1, \ldots, k \) such that the dipath is \( 1 \rightarrow 2 \rightarrow \ldots \rightarrow k \) (next page)
Case one: \( D \) has a path \( \overline{v_1v_2} \).
Then the Hamiltonian circuit is
\[
\overline{v_1v_2} \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n.
\]

Case two: Assume that \( D \) has a bridge \( v \rightarrow v_{e+1} \). The Hamiltonian circuit is
\[
v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{e-1} \rightarrow v_e \rightarrow v_{e+1}.
\]

Case three: Case one and case two do not hold.
So \( \overline{v_1v_{e+1}} \) and \( \overline{v_{e+1}v_e} \) followed.
Then there must exists \( 1 \leq i \leq e \)
such that \( v_{i-1} \rightarrow v_i \)
\[
v_i \rightarrow v_{e+1}.
\]
Then the H. C. is
\[
v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{i-1} \rightarrow v_i \rightarrow v_{e+1} \rightarrow v_e \rightarrow \cdots \rightarrow v_n.
\]

3. Because all edges in \( K_n \) are same (up to graph \( K_n \) iso), let \( k(n) \) be the number of spanning trees of \( K_n \) that include a given edge. Let us count the number of edges in all spanning trees of \( K_n \). Each spanning tree has \( (n-1) \) edges.
The number of spanning trees by Cayley's theorem is \( n^{n-2} \). So the total number of edges is \( \frac{n(n-1)}{2} \). So the total number of edges is \( \frac{n(n-1)}{2} \).
3. Hence the total number of edges in all spanning trees is \( \frac{n(n-1)}{2} \).

Here \( (n-1) \) \( n^{n-2} = n(n-1) \) \( k(n) \)

So \( k(n) = 2n^{n-3} \)

The number of spanning trees of \( K_n \) not containing a given edge is

\[ n^{n-2} - k(n) = n^{n-2} - 2n^{n-3} = (n-2)n^{n-3} \]

4. Take the symmetric difference \( M \Delta N = L \)

It is not empty set since \( M \neq N \).

The symmetric difference \( L \) includes a subgraph which consists of even cycles and paths: \((C_1, U_1 U_2 K_1) U (P_1 U_1 U_2 P_2)\)

Each even cycle \( C \) has \( \frac{n}{2} \) edges in \( M \) and \( \frac{n}{2} \) edges in \( N \).

Let \( M_1 = M \setminus L \), \( N_1 = N \setminus L \)

and \( M_2 = M \setminus M_1 = N_2 = N \setminus N_1 \).

As \( |M_1| > |N_1| \) it follows that \( |M_2| > |N_2| \).

Hence there must be a path \( P_1 \) that contains more edges in \( M \) than in \( N \).

This is a augmenting path.

\[ P_1 \]

Let \( N' = (M_1 U_2 U_3 \ldots U_{2k-1} U_{k+2}) U U_{k+2, k} \]

\( U_2, U_3, U_4, \ldots, U_{2k-1}, U_{k+2} \).

So \( |N'| = |N| + 1 \).
\[ M' = (M \setminus \cup_{i=1}^{k} U_{i,2i} \cup_{i=1}^{k} U_{2i-2,2i-1}) \cup U_{2k,2k+1} \]

So \( |M'| = |M| - 1 \).

2. \( M \subseteq E \)

- \( M = \{ U_{1,2}, U_{3,4}, \ldots, U_{2k-1,2k} \} \)

- \( k \) edges in \( E \) with no common vertices.

1. \( \text{max} |M| = \alpha''(G) \)

2. \( \mathcal{Q} \subseteq E \) edge cover:

   - for every vertex \( u \in V \) \( \exists \ uv \in \mathcal{Q} \).

1. \( \text{min} |\mathcal{Q}| = \beta'(G) \)

b. First \( |M'| + |\mathcal{Q}| \geq n/6 \).

Each vertex of \( \mathcal{Q} \) covers at least on vertex in \( M' \). The best efficient way is that for \( \mathcal{Q} \) is that each edge in \( \mathcal{Q} \) covers two vertices in \( M' \).

Hence \( |M'\left\vert + |\mathcal{Q}| \geq n/6 \). \)

Note: Hence \( \alpha''(G) + \beta'(G) \geq n/6 \).

Now we want to show that equality holds for some \( M \) and \( \mathcal{Q} \). First choose \( M \) maximum \( \alpha'' \) match. \( |M| = \alpha''(G) \).
Now take a set \( V(M) \) be all the vertices in \( M \). So \( |V(M)| = 2|M| \).

Consider the induced subgraph \( G(V \setminus V(M)) \).

It reduces to a number of connected components \( G_1 = (V, E_1), \ldots, G_k = (V, E_k) \) for each \( G_i \) take a spanning tree \( T_i \).

First observe that the diameter of \( T_i \) is at most 2. Otherwise, we could add it to the \( M \). So \( M \) is not a matching!

Take \( Q = M \cup (S_1 \cup \cdots \cup S_k) \),

\[ |M| + |Q| = w(G). \]

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