Chapter 8

2. Show that \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) has seven subgroups of order 2.

Solution: We can list the elements of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) explicitly, and there are 8 of them:

\[
(0,0,0), (1,0,0), (0,1,0), (1,1,0), (0,0,1), (1,0,1), (0,1,1), (1,1,1).
\]

Now \( [0] + [0] = [1] + [1] = [0] \) in \( \mathbb{Z}_2 \), and the direct sum construction is defined with operation:

\[
(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).
\]

Thus, if we take any \( a = (x, y, z) \) element in \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) then we must have \( a^2 = (x, y, z)(x, y, z) = (x + x, y + y, z + z) = ([0], [0], [0]) \), the identity of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

There are seven elements of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) of order 2 (every element except \( e \)), and for each such \( a \) there is a subgroup of order 2, namely \( \{e, a\} \). This gives seven different subgroups.

However, this is all of the subgroups of order 2, since a subgroup of order 2 has \( e \) and one other element.

4. Show that \( G \oplus H \) is abelian if and only if \( G \) and \( H \) are abelian.

Solution: Suppose that \( G \) and \( H \) are abelian, and that \( (g_1, h_1), (g_2, h_2) \in G \oplus H \). Then

\[
(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)
\]

\[
= (g_2g_1, h_2h_1) \text{ since } G \text{ and } H \text{ are abelian},
\]

\[
= (g_2, h_2)(g_1, h_1).
\]

Therefore \( G \oplus H \) is abelian.

Conversely, suppose \( G \oplus H \) is abelian and let \( g_1, g_2 \in G, h_1, h_2 \in H \). Then

\[
(g_1g_2, h_1h_2) = (g_1, h_1)(g_2, h_2)
\]

\[
= (g_2, h_2)(g_1, h_1) \text{ since } G \oplus H \text{ is abelian},
\]

\[
= (g_2g_1, h_2h_1).
\]

Therefore, \( g_1g_2 = g_2g_1 \), and \( G \) is abelian, and \( h_1h_2 = h_2h_1 \), so \( H \) is abelian.

12. The dihedral group \( D_n \) of order \( 2n \) (\( n \geq 3 \)) has a subgroup of \( n \) rotations and a subgroup of order 2. Explain why \( D_n \) cannot be isomorphic to the external direct product of two such groups.

Solution: The rotation subgroup of \( D_n \) is abelian (we've seen this in class many times), and the subgroup of order 2 is abelian (since we know that the only group of order 2, up to isomorphism, is the cyclic group of order 2).

Therefore, the direct product of the rotation subgroup and a group of order 2 is abelian, by Question 4. But if \( n \geq 3 \), then \( D_n \) is not abelian. Therefore, \( D_n \) cannot be a direct product of these two groups.
Chapter 9

4 Let \( H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R}, ad \neq 0 \right\} \). Is \( H \) a normal subgroup of \( \text{GL}(2, \mathbb{R}) \)?

Solution:
Let \( x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \text{GL}(2, \mathbb{R}) \). Then
\[
x^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

Let \( y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H \).

We calculate:
\[
x^{-1}yx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\]

Therefore, \( x^{-1}yx \notin H \). But, if \( H \) were normal then \( x^{-1}Hx \subseteq H \) for any \( x \in \text{GL}(2, \mathbb{R}) \). So \( H \) is not normal in \( \text{GL}(2, \mathbb{R}) \).

10 Prove that a factor group of a cyclic group is cyclic.

Solution: Suppose that \( G = \langle a \rangle \) and that \( H \trianglelefteq G \). An element of \( G/H \) has the form \( gH \) for some \( g \in H \). Each element \( g \) can be written as \( a^k \) for some \( k \). Now
\[
a^kH = (aH)^k,
\]
(as can be seen by an easy inductive proof, and the definition of the product in \( G/H \)).

Therefore \( G/H = \langle aH \rangle \) is cyclic, as required.

14 What is the order of the element \( 14 + \langle 8 \rangle \) in the factor group \( \mathbb{Z}_{24}/\langle 8 \rangle \).

Solution: \( \langle 8 \rangle \) has order 3 in \( \mathbb{Z}_{24} \), so \( \mathbb{Z}_{24}/\langle 8 \rangle \) is cyclic of order 8, generated by \( [1] + \langle 8 \rangle \). Now \( \langle 8 \rangle = \{0, [8], [16]\} \subset \mathbb{Z}_{24} \). We can calculate explicitly:
\[
[14] + \langle 8 \rangle \neq \langle 8 \rangle \text{ (since } 14 \notin \langle 8 \rangle \)
\]
\[
([14] + \langle 8 \rangle)^2 = [28] + \langle 8 \rangle
\]
\[
\quad = [4] + \langle 8 \rangle \neq \langle 8 \rangle
\]
\[
([14] + \langle 8 \rangle)^3 = [42] + \langle 8 \rangle
\]
\[
\quad = [18] + \langle 8 \rangle \neq \langle 8 \rangle
\]
\[
([14] + \langle 8 \rangle)^4 = [56]\langle 8 \rangle
\]
\[
\quad = [8] + \langle 8 \rangle = \langle 8 \rangle
\]

Therefore, \([14] + \langle 8 \rangle \) has order 4 in \( \mathbb{Z}_{24}/\langle 8 \rangle \).