1 Elementary number theory

We assume the existence of the natural numbers

\[ \mathbb{N} = \{1, 2, 3, \ldots\} \]

and the integers

\[ \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \]

along with their most basic arithmetical and ordering properties.

For example, we assume the truth of statements such as

- "For all integers \(a\) and \(b\), both \(a + b\) and \(ab\) are integers."
- "\(0 < 1\)"
- "For all integers \(a\) and \(b\) we have \(a + b = b + a\)."
- "For any pair of integers \(a\) and \(b\), exactly one of the following is true: \(a = b\), \(a < b\) or \(a > b\)."

**Definition 1.** Let \(a\) and \(b\) be integers. We say \(a\) divides \(b\) if there exists an integer \(k\) such that \(ka = b\).\(^1\)

**Notation 2.** The notation \(a \mid b\) means “\(a\) divides \(b\).”

**Proposition 3.** Let \(a\), \(b\) and \(c\) be integers. If \(a \mid b\) and \(a \mid c\) then \(a \mid (b + c)\).

**Proposition 4.** Let \(a\), \(b\) and \(c\) be integers. If \(a \mid b\) and \(a \mid c\) then \(a \mid (b - c)\).

**Conjecture 5.** Let \(a\), \(b\) and \(c\) be integers. If \(a \mid (b + c)\), then \(a \mid b\) or \(a \mid c\).

**Proposition 6.** Let \(a\), \(b\) and \(c\) be integers. If \(a \mid b\) then \(a \mid bc\).

**Proposition 7.** Let \(d\), \(a\), \(b\), \(x\) and \(y\) be integers. If \(d \mid a\) and \(d \mid b\), then \(d \mid (ax + by)\).

**Proposition 8.** Let \(a\), \(b\) and \(c\) be integers. If \(a \mid b\) and \(b \mid c\) then \(a \mid c\).

**Proposition 9.** \(2 \nmid 1\).

\(^{1}\text{We will prove later that “}\(a\) divides \(b\)\)” is equivalent to “the rational number } \frac{b}{a} \text{ is an integer.” However, that is not our definition. We don’t know what a rational number is yet.}