5 Worksheet: The Real Numbers, II

The last proposition on the previous worksheet was

**Proposition 11.** For every \( a \) and \( b \) in \( \mathbb{R} \), \((-a)(-b) = ab\).

**Corollary 12.** \((-1)(-1) = (1)(1) = 1\).

**Task 12.1.** Write your own Propositions 13 and 14. They should be things which are true about real numbers, are not stated as axioms and that you think should be easy to prove. You do not have to provide a proof of your statements, or even know how a formal proof should go.

**Proposition 15.** \( 0 < 1 \).

**Proposition 16.** For all \( a \) and \( b \) in \( \mathbb{R} \), if \( a \neq 0 \) and \( b \neq 0 \) then \((ab)^{-1} = a^{-1}b^{-1}\).

**Proposition 17.** For every \( a \) and \( b \) in \( \mathbb{R} \), \(- (a + b) = -a + (-b)\).

**Definition 18.** We define subtraction as a function \(- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) by \( a - b = a + (-b)\).

**Question 19.** Is the operation of subtraction (as defined above) well-defined? Why or why not?

**Proposition 20.** The operation of subtraction is not commutative, i.e. it is not true that for all real numbers \( a \) and \( b \), \( a - b = b - a \).

**Notation 21.** Let \( a \) be a real number. The number \( a^2 \) is defined to be \( a.a \).

**Proposition 22.** For all \( x \) and \( a \) in \( \mathbb{R} \), \((x + a)(x - a) = x^2 - a^2\).

**Remark 23.** Reading the axioms on the first worksheet, we see that the only axioms which involve \( < \) are about comparing a real number to \( 0 \). Moreover, they all talk about \( 0 < x \) for some number \( x \). We do not yet know what it means to say that a real number \( a \) is less than another real number \( b \). Let’s define that.

**Definition 24.** Let \( a \) and \( b \) be real numbers. We say that \( a \) is less than \( b \) if \( 0 < b - a \).

**Notation 25.** If \( a < b \) then we would also like to say \( b \) is greater than \( a \). We will say this and we will write \( b > a \).

**Proposition 26.** For every \( a \) in \( \mathbb{R} \), if \( a > 0 \) then \(-a < 0 \). Also, if \( a < 0 \) then \(-a > 0 \).

**Proposition 27.** For all \( a \) and \( b \) in \( \mathbb{R} \), exactly one of the following is true: \( a < b \), \( a = b \) or \( a > b \).