Math 502
Problem Set #6
Due November 18

Problem 1: Prove there is \( e \in \mathbb{N} \) such that \( \phi_e(x) = e \) for all \( x \in \mathbb{N} \).

Problem 2: Set \( \text{Ext} := \{ e \in \mathbb{N} : \phi_e \text{ is extendible to a total computable function.} \} \). Prove that:

(a) \( \text{Ext} \) is \( \Sigma_3 \).
(b) \( \text{Ext} \neq \mathbb{N} \).

Problem 3: Show that if \( A \leq_m B \) and \( A \) is productive, then \( B \) is productive.

Problem 4: Show that \( \text{Tot} \) and \( \neg \text{Tot} \) are both productive.

Problem 5: Show (without using Post’s theorem) that \( \neg \text{Tot} \) is r.e. in \( K \).

Problem 6: Find \( A \in 0' \) such that neither \( A \) nor \( \neg A \) are r.e.

Problem 7: Give examples of sets \( A, B, C \subseteq \mathbb{N} \) such that \( A \) is r.e. in \( B \) and \( B \) is r.e. in \( C \) but \( A \) is not r.e. in \( C \).

Given any set \( A \subseteq \mathbb{N} \), one can define the relativized arithmetic hierarchy with respect to \( A \). For example, the \( \Sigma_1^A \) sets are just the sets that are r.e. in \( A \). Given this, one can formulate and prove a relativized version of Post’s Theorem. In the next problem, you may use this relativized version of Post’s Theorem.

Problem 8: A degree \( a \) is said to be low if \( a \leq 0' \) but \( a' = 0' \). A set \( A \) is low if its Turing degree is low. For \( A \leq_T \emptyset' \), prove that the following are equivalent:

1. \( A \) is low;
2. \( \Sigma_1^A \subseteq \Pi_2 \);
3. \( A' \leq_1 \neg \emptyset'' \).