Axiomatizing changing conceptions of the geometric continuuum II: Archimedes - Descartes-Hilbert -Tarski

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In Part I [Bal14a], we defined the notion of a modest complete descriptive axiomatization and showed that HP5 and EG are such axiomatizations of Euclid's polygonal geometry and Euclidean circle geometry[1]. In this paper we argue: 1) Tarski's axiom set $\mathcal{E}^2$ is a modest complete descriptive axiomatization of Cartesian geometry (Section 2; 2) the theories $EG_{\pi,C,A}$ and $\mathcal{E}^2_{\pi,C,A}$ are modest complete descriptive axiomatizations of Euclidean circle geometry and Cartesian geometry, respectively when extended by formulas computing the area and circumference of a circle (Section 3); and 3) that Hilbert's system in the Grundlagen is an immodest axiomatization of any of these geometries. As part of the last claim (Section 4), we analyze the role of the Archimdedean postulate in the Grundlagen, trace the intricate relationship between alternative formulations of 'Dedekind completeness', and exhibit many other categorical axiomatizations of related geometries.

1 Background and Definitions

In [Bal14a], we expounded the following historical description.

Remark 1.1 (Background)**.** Euclid founds his theory of area (of circles and polygons) on Eudoxus' theory of proportion and thus (implicitly) on the axiom of Archimedes.

Hilbert shows any 'Hilbert plane' interprets a field and recovers Euclid's theory of polygons in a first order theory.

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[1]These axioms are described in Notation 1.3; these data sets are described as specific sets of propositions from the Elements in [Bal14a].

The Greeks and Descartes dealt only with geometric objects. The Greeks regarded multiplication as an operation from line segments to plane figures. Descartes interpreted it as an operation from line segments to line segments. Only in the late 19th century, is multiplication regarded as an operation on points (that is 'numbers' in the coordinatizing field).

We built in [Bal14a] on Detlefsen's notion of complete descriptive axiomatization and defined a *modest complete descriptive axiomatization* of a data set Σ to be collection of sentences that imply all the sentences in Σ and 'not too many more'. A data set is a collection of propositions about a mathematical topic that are accepted at a given point in time. Of course, there will be further results proved about this topic. But if this set of axioms introduces essentially new concepts to the area or, even worse, contradicts the understanding of the original era, we deem the axiomatization immodest. We illustrate these definitions by specific axiomatizations of various areas of geometry that we now describe.

We formulate our system in a two-sorted vocabulary τ chosen to make the Euclidean axioms (either as in Euclid or Hilbert) easily translatable into first order logic. This vocabulary includes unary predicates for points and lines, a binary incidence relation, a ternary betweenness relation, a quaternary relation for line congruence and a 6-ary relation for angle congruence. We need one additional first order postulate[2] beyond those in [Hil71].

Postulate 1.2. Circle Intersection Postulate *If from points A and B, circles with radius AC and BD are drawn such that one circle contains points both in the interior and in the exterior of the other, then they intersect in two points, on opposite sides of AB.*

Notation 1.3. We follow Hartshorne[Har00] in the following nomenclature.

A *Hilbert plane* is any model of Hilbert's incidence, betweenness[3], and congruence axioms[4]. We denote this axiom as HP and write HP5 for these axioms plus the parallel postulate.

By the *axioms for Euclidean geometry* we mean HP5 and in addition the circle-circle intersection postulate 1.2. We will abbreviate this axiom set[5] as EG. By

[2]Moore suggests in [Moo88] that Hilbert may have added the completeness axiom to the second edition specifically because Sommer in his review of the first edition pointed out it did not prove the line-circle intersection principle.

[3]These include Pasch's axiom (B4 of [Har00]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[4]These axioms are equivalent to the common notions of Euclid and Postulates I-V augmented by one triangle congruence postulate, usually taken as SAS since the 'proof' of SAS is where Euclid makes illegitimate use of the superposition principle.

[5]In the vocabulary here, there is a natural translation of Euclid's axioms into first order statements. The construction axioms have to be viewed as 'for all– there exist' sentences. The axiom of Archimedes as discussed below is of course not first order. We write Euclid's axioms for those in the original [Euc56] vrs (first order) axioms for Euclidean geometry, EG. Note that EG is equivalent to (i.e. has the same models) as the system laid out in Avigad et al [ADM09], namely, planes over fields where every positive element as a square root). The latter system builds the use of diagrams into the proof rules.

definition, a *Euclidean plane* is a model of EG: Euclidean geometry.

We write $\mathcal{E}^2$ for Tarski's geometrical axiomatization of the plane over a real closed field (RCF) (Theorem 2.1).

2 From Descartes to Tarski

It is not our intent to give a detailed account of Descartes' impact on geometry. We want to bring out the changes from the Euclidean to the Cartesian data set. For our purposes, the most important is to explicitly (on page 1 of [Des54]) define the multiplication of line segments to give a line segment which breaks with Greek tradition[6]. And later on the same page to announce constructions for the extraction of nth roots for all n. The second of these cannot be done in EG, since it is satisfied in the field which has solutions for all quadratic equations but not those of odd degree[7].

Marco Panza [Pan11] formulates in terms of ontology a key observation,

> The first point concerns what I mean by 'Euclid's geometry'. This is the theory expounded in the first six books of the Elements and in the Data. To be more precise, I call it 'Euclids plane geometry', or EPG, for short. It is not a formal theory in the modern sense, and, a fortiori, it is not, then, a deductive closure of a set of axioms. Hence, it is not a closed system, in the modern logical sense of this term. Still, it is no[8] more a simple collection of results, nor a mere general insight. It is rather a well-framed system, endowed with a codified language, some basic assumptions, and relatively precise deductive rules. And this system is also closed, in another sense ([Jul64] 311-312), since it has sharp-cut limits fixed by its language, its basic assumptions, and its deductive rules. In what follows, especially in section 1, I shall better account for some of these limits, namely for those relative to its ontology. More specifically, I shall describe this ontology as being composed of objects available within this system, rather than objects which are required or purported to exist by force of the assumptions that this system is based on and of the results proved within it. This makes EPG radically different from modern mathematical theories (both formal and informal). One of my claims is that Descartes geometry partially reflects this feature of EPG.

In our context we interpret 'composed of objects available within this system' model theoretically as the existence of certain starting points and the closure of each model of the system under admitted constructions.

[6]His proof is still based on Eudoxus.
[7]See section 12 of [Har00].
[8]There appears to be a typo. Probably 'more a" should be deleted. jb

We take Panza's 'open' system to refer to Descartes' 'linked constructions'[9] considered by Descartes which greatly extend the ruler and compass licensed in EG. Descartes endorses such 'mechanical' constructions as the duplication of the cubic as geometric. According to Molland (page 38 of [Mol76]) "Descartes held the possibility of representing a curve by an equation (specification by property)" to be equivalent to its "being constructible in terms of the determinate motion criterion (specification by genesis)". But as Crippa points out (page 153 of [Cri14a]) Descartes did not prove this equivalence; there is some controversy as to whether the 1876 work of Kempe solves the precise problem. Descartes rejects as non-geometric any method for quadrature of the circle. See page 48 of [Des54] for his classification of problems by degree. Unlike Euclid, Descartes does not develop his theory axiomatically. But an advantage of the 'descriptive axiomatization' rubric is that we can take as data the theorems of Descartes' geometry.

For our purpose we take the common identification of Cartesian geometry with "real" algebraic geometry: the study of polynomial equalities and inequalities in the theory of real closed fields. To justify this geometry we adapt Tarski's 'elementary geometry'. This move makes a significant conceptual step away from Descartes whose constructions were on segments and who did not regard a line as a set of points while Tarski's axiom are given entirely formally in a one sorted language of relations on points. In our modern understanding of an axiom set the translation is routine. From Tarski [Tar59] we get:

Theorem 2.1. *Tarski [Tar59] gives a theory equivalent to the following system of axioms $\mathcal{E}^2$. It is first order complete for the vocabulary τ.*

1. *Euclidean geometry*
2. *Either of the following two sets of axioms which are equivalent over i)*
   (a) *An infinite set of axioms declaring that every polynomial of odd-degree has a root.*
   (b) *The axiom schema of continuity described just below.*

The connection with Dedekind's approach is seen by Tarski's actual formulation as in [GT99]; the first order completeness of the theory is imposed by an **Axiom Schema of Continuity** - a definable version of Dedekind cuts:

$$(\exists a)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(axy)] \rightarrow (\exists b)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(xby)],$$

where $B(x, y, z)$ is the predicate representing y is between x and z, α, β are first-order formulas, the first of which does not contain any free occurrences of a, b, y nor the second any free occurrences of a, b, x. This schema allows the solution of odd degree polynomials.

[9]The types of constructions allowed are analyzed in detail in Section 1.2 of [Pan11] and the distinctions with the Cartesian view in Section 3. See also [Bos01].

Remark 2.2 (Gödel completeness)**.** In Detlefsen's terminology we have found a Gödel complete axiomatization of (in our terminology Cartesian) plane geometry. This guarantees that if we keep the vocabulary and continue to accept the same data set no axiomatization can account for more of the data. There are certainly open problems in plane geometry [KW91]. But however they are solved the proof will be formalizable in $\mathcal{E}^2$. Of course, more perspicuous axiomatizations may be found. Or one may discover the entire subject is better viewed as an example in a more general context.

In the case at hand, however, there are more specific reasons for accepting the geometry over real closed fields as 'the best' descriptive axiomatization. It is the only one which is decidable and 'constructively justifiable' (See Theorem 3.3.).

Remark 2.3 (Undecidability and Consistency)**.** Ziegler [Zie82] has shown that every nontrivial finitely axiomatized subtheory of RCF[10] is not decidable.

Thus both to more closely approximate the Dedekind continuum and to obtain decidability we restrict to planes over RCF and thus to Tarski's $\mathcal{E}^2$ [GT99]. Of course, another crucial contribution of Descartes is coordinate geometry. Tarski provides a converse; his interpretation of the plane into the coordinatizing line [Tar51] underlies our smudging of the study of the 'geometry continuum' with axiomatizations of 'geometry'. The biinterpretability[11] between RCF and the theory of all planes over real closed fields yields the decidability of $\mathcal{E}^2$. The crucial fact that makes decidability possible is that the natural numbers are *not first order definable* in the real field. The geometry can represent multiplication as repeated addition in the sense of a module over a $\mathbb{Z}$ but not with the full ring structure.

3 Archimedes: π, circumference and area of circles

The geometry over a Euclidean field (every positive number has a square root) may have no straight line segment of length π, since the model containing only the constructible real numbers does not contain π. We extend by adding π each geometry EG and $\mathcal{E}^2$ and *write T when discussing results that apply to either*. We want to find a theory which proves the circumference and area formulas for circles. Our approach is to extend the theory EG so as to guarantee that there is a point in every model which behaves as π does. In this section we will show that in this extended theory there is a mapping assigning a straight line segment to the circumference of each circle. This goal definitely diverges from a 'Greek' data set and is orthogonal to the axiomatization of Cartesian geometry in Section 2. Given that the entire project is modern, we give the arguments entirely in modern style.

[10] RCF abbreviates 'real closed field'; these are the ordered fields such that every positive element has a square root and every odd degree polynomial has at least one root. The theory is complete and recursively axiomatized so decidable. By nontrivial subtheory, I mean one satisfied by one of $\mathbb{C}, \Re$, or a p-adic field Q_p. For the context of Ziegler's result and Tarski's quantifier elimination in computer science see [Mak13].

[11] Tarski proves the equivalence of geometries over real closed fields with his axiom set in [Tar59]. He calls the theory elementary geometry, $\mathcal{E}^2$.

Euclid differs from the modern approach in that the sequences constructed in the study of magnitudes in the *Elements* are of geometric objects, not (even real) numbers. (Implicitly) using Archimedes axiom, Euclid proves (XII.2) that the area of a circle is proportional to the square of the diameter. In a modern account, as we saw in [Bal14a], when we have interpreted segments Oa as representing the number a, we must identify the proportionality constant and verify that it represents a point in any model of the theory[12]. This shift in interpretation drives the rest of this section. We search first for the solution of a specific problem: is π in the underlying field?

We named this section after Archimedes for two reasons. He is the first (Measurement of a Circle in [Arc97]) to prove the circumference of a circle is proportional to the diameter and begin the approximation of the proportionality constant (which wasn't named for another 2000 years). Secondly, his axiom is used not only in his work but implicitly[13] by Euclid in proving the area of a circle is proportional to the square of the diameter. By beginning to calculate approximations of π, Archimedes is moving towards the treatment of π as a number. The validation in the theory $\mathcal{E}^2_{\pi,C,A}$ below of the formulas $A = \pi r^2$ and $C = \pi d$ are answering the questions of Hilbert and Dedekind not questions of Euclid or even Archimedes. But the theory EG_π is closer to the Greek origins than to Hilbert's second order axioms.

Dicta 3.1. *Constants 2* Recall (Section 4.4) of [Bal14a]) that closing a plane under ruler and compass constructions corresponds to closing the coordinatizing ordered field under square roots of positive numbers. Having named an arbitrary points as $0, 1$, each element of the maximal real quadratic extension of the rational field (the surd field F_s) is denoted by a term $t(x, 0, 1)$ built from the field operations and $\sqrt{\ }$.

In Definition 3.2 we name a further single constant π. But the effect is much different than naming 0 and 1). The new axioms specify the place of π in the ordering of the definable points of the model. So the data set is seriously extended.

Now that we have established that each model of T has the surd field F_s embeddable in the field definable in any line of the model, we can interpret the Greek theory of proportionality in terms of cuts. Each pair of proportional pairs of magnitudes determines a cut (E.g. page 33-34 of [Coo63]). The non-first order postulates of Hilbert play complementary roles. The Archimedean axiom is minimizing; each cut is realized by at most one point so each model has cardinality at most $2^{\aleph_0}$. The Veronese postulate (See Footnote 20.) or Dedekind's postulate is maximizing; each cut is realized, In Hilbert's version, the set of realizations could have arbitrary cardinality.

Definition 3.2 (Axioms for π). *1. Add to the vocabulary a new constant symbol π. Let i_n (c_n) be the perimeter of a regular $3 * 2^n$-gon inscribed*[14] *(circumscribed)*

[12] For this reason, Archimedes needs only his postulate while Hilbert also needs Dedekind's postulate to prove the circumference formula.

[13] We adopt this rather common interpretation. But Borzacchini [Bor06] suggests that Archimedes real innovation is to require an additional hypothesis to compare straight segments with curved ones.

[14] I thank Craig Smorynski for pointing out that is not so obvious that that the perimeter of an inscribed n-gon is monotonic in n and reminding me that Archimedes started with a hexagon and doubled the number of sides at each step.

in a circle of radius 1*. Let* $\Sigma(\pi)$ *be the collection of sentences (i.e. type*[15]*)*

$$i_n < 2\pi < c_n$$

for $n < \omega$*.*

2. *We extend both* EG *and* $\mathcal{E}^2$*.*
   (a) EG_π *denotes deductive closure in the vocabulary* τ *augmented by constant symbols* $0, 1, \pi$ *of EG and* $\Sigma(\pi)$*.*
   (b) $\mathcal{E}^2_\pi$ *is formed by adding* $\Sigma(\pi)$ *to* $\mathcal{E}^2$ *and taking the deductive closure.*

For the second proposition in the next theorem we use some modern model theory. A first order theory T for a vocabulary including a binary relation $<$ is *o-minimal* if every 1-ary formula is equivalent in T to a Boolean combination of equalities and inequalities [dD99]. Anachronistically, the o-minimality of the reals is a main conclusion of Tarski in [Tar31].

The two theories we have constructed have quite different properties.

Theorem 3.3. *1.* EG_π *is a consistent but incomplete theory. It is not finitely axiomatizable.*

2. $\mathcal{E}^2$ *and* $\mathcal{E}^2_\pi$ *are complete decidable o-minimal theories. It is provably consistent in primitive recursive arithmetic (PRA).*

Proof. 1) A model of EG_π is given by closing $F_s \cup \{\pi\} \subseteq \Re$ under ruler and compass constructions. To see it is not finitely axiomatizable, for any finite subset Σ_0 of Σ choose a real algebraic number p satisfying Σ_0; close $F_s \cup \{p\} \subseteq \Re$ under constructibility to get a model of EG which is not a model of EG_π.

2) We have established that there are well-defined field operations on the line through 01. By Tarski, the theory of this real closed field is complete. The field is bi-interpretable with the plane [Tar51] so the theory of the geometry T is complete as well. Further by Tarski, the field is o-minimal. The type over the empty set of any point on the line is determined by its position in the linear ordering of the surd subfield F_s. Each i_n, c_n is an element of the field F_0. This position in the linear order of 2π in the linear order on the line through 01 is given by Σ. Thus $T \cup \Sigma$ is complete.

The proof that consistency is provable in PRA ([Bal14b]) uses different methods. The arguments of ([Sim09] IX.3.18) show the theory of real closed fields is provably consistent in WKL_0; Since the compactness theorem is also provable in WKL_0 this allows us to extend to the consistency of $\mathcal{E}^2_\pi$. And the consistency passes to PRA since WKL_0 is conservative over PRA for π^0_1-sentences. $\square_{3.3}$

[15]Let $A \subset M \models T$. A type over A is a set of formulas $\phi(\mathbf{x}, \boldsymbol{a})$ where $\mathbf{x}$, $(\boldsymbol{a})$ is a finite sequence of variables (constants from A) that is consistent with T. It is over the empty set if the elements of A are definable without parameters in T (e.g. the constructible numbers). Here we take T as EG.

Dicta 3.4 (Definitions or Postulates I)**.** In Definition 2.5 of [Bal14b] we extend the ordering on segments by adding the lengths of 'bent lines' and arcs of circles to the domain. Two approaches[16] to this step are a) our approach to introduce an explicit but inductive definition or b) add a new predicate to the vocabulary and new axioms specifying it behavior. The second alternative reflects in a way the trope that Hilbert's axioms are *implicit definitions*. But this choice is not available for the initial axiomatization. It is only because we have already established a certain amount of geometric vocabulary that we can take choice a). Crucially the definition of bent lines (and thus the perimeter of certain polygons) is not a single definition but a schema of formulas ϕ_n defining the property for each n. Our definition is more restricted than Archimedes who really foresees the notion of bounded variation.

In the following, T denotes either EG or $\mathcal{E}^2$. To argue that π, as implicitly defined by the theory T_π, serves its geometric purpose, we add a new unary function symbol C mapping our fixed line to itself and (Definition 2.6 of [Bal14b]) satisfying a scheme asserting that for each n, $C(r)$ is between (in the sense of the last paragraph) the perimeter of a regular inscribed n-gon and a regular circumscribed n-gon of a circle with radius r. Any function $C(r)$ satisfying these conditions is called *a circumference function*; we call $C(r)$ the *circumference* of a circle with radius r.

The justification of the area formula is simpler because we do not need the step of extending the ordering on straight lines to arcs. Following Hilbert we noted in Section 4.6 of [Bal14a] that if one polygon is included in another it has a smaller area. So the approximation is less of an issue than for circumference and we just need the calculation in the surd field that estimates of π computed from the standpoints of area and circumference are in the same cut in F_s.

Lemma 3.5. *Let, for $n < \omega$, I_n and C_n denote the area of the regular 3×2^n-gon inscribed or circumscribing the unit circle. Then T_π proves each of the sentences $I_n < \pi < C_n$.*

We extend the theory to include a definition of $C(r)$ and $A(r)$.

Definition 3.6.
1. *The theory $T_{\pi,C}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory T_π obtained by the explicit definition $C(r) = 2\pi r$.*
2. *The theory $T_{\pi,C,A}$ is the extension of the $\tau \cup \{0, 1, \pi, C\}$-theory $T_{\pi,C}$, obtained by the explicit definition $A(r) = \pi r^2$.*

As an extension by explicit definition, $\mathcal{E}^2_{\pi,C}$ is complete and o-minimal. Our definition of T_π then makes the following metatheorem immediate.

Theorem 3.7. *In $T^2_{\pi,C}$, $C(r) = 2\pi r$ is a circumference function (i.e. satisfies all the conditions ι_n and γ_n) and in $T^2_{\pi,C,A}$, $A(r) = \pi r^2$ is an area function.*

[16] More specifically, we could define $<$ on the extended domain or we could add an $<^*$ to the vocabulary and postulate that $< *$ extends $<$ and satisfies the properties of the definition.

The proof for circumference shows that for each r there is an $s \in \mathcal{S}$ whose length, $2\pi r$ is less than the perimeters of all inscribed polygons and greater that those of the inscribed polygons. We can verify that by choosing n large enough we can make i_n and c_n as close together as we like (more precisely, for given m differ by $< 1/m$). In phrasing this sentence I follow Heath's description[17] of Archimedes statements, "But he follows the cautious method to which the Greeks always adhered; he never says that given curve or surface is the *limiting form* of the inscribed or circumscribed figure; all that he asserts is that we can approach the curve or surface as nearly was we please".

We have *not* established the arc length claim for each arc in $\mathcal{C}_r$ for even one r. We will accomplish that task in Lemma 3.10.

In an Archimedean field there is a unique interpretation of π and thus a unique choice for a circumference function with respect to the vocabulary without the constant π. By adding the constant π to the vocabulary we get a formula which satisfies the conditions in every model. But in a non-Archimedean model, any point in the monad of $2\pi r$ would equally well fit our condition for being the circumference.

We have extended our descriptively complete axiomatization from the polygonal geometry of Hilbert's first order axioms (HP5) to Euclid's results on circles and beyond. Euclid doesn't deal with arc length at all and we have assigned straight line segments to both the circumference and area of a circle. So this would not qualify as a modest axiomatization of Greek geometry but only of the modern understanding of these formulas. This distinction is not a problem for the notion of descriptive axiomatization. The facts are sentences. The formulas for circumference and are not the same sentences as the Euclid/Archimedes statement in terms of proportions.

We could however better argue that $EG_{\pi,A}$ is modest axiomatization of Euclid, than $EG_{\pi,C,A}$ given the absence of arc length from Euclid (and the presence of VI.1).

Now we extend our goals to find a modest descriptive axiomatization of the modern conception of *angle measure*. This will illuminate the delicate dependence of descriptive axiomatization on the vocabulary in which the facts are expressed.

Dedekind (page 37-38) observes that what we would now call the real closed field with domain the field of real algebraic numbers is 'discontinuous everywhere' but 'all constructions that occur in Euclid's elements can ... be just as accurately effected as in a perfectly continuous space'. Strictly speaking, for *constructions* this is correct. But the proportionality constant between a circle and its circumference π is absent, so, even more, not both a straight line segment of the same length as the circumference and the diameter are in the model We want to find a theory which proves the circumference and area formulas for circles and countable models of the geometry over RCF, where 'arc length behaves properly'.

Mueller (page 236 of [Mue06]) makes an important point distinguishing the

[17] Archimedes, Men of Science [Hea11], introduction Kindle location 393.

use of cuts in Euclid/Eudoxus from that in Dedekind.

> One might say that in applications of the method of exhaustion the limit is given and the problem is to determine a certain kind of sequence converging to it, ...Since, in the *Elements* the limit always has a simple description, the construction of the sequence can be done within the bounds of elementary geometry; and the question of constructing a sequence for any given arbitrary limit never arises.

This distinction can be expressed in another way. We speak of the method of Eudoxus: a technique to solve certain problems, which are specified in each application. In contrast, Dedekind's postulate provides a solution for $2^{\aleph_0}$ problems.

Descartes eschews the idea that there can be a ratio between a straight line segment and a curve. As [Cri14b] writes, "Descartes[18] excludes the exact knowability of the ratio between straight and curvilinear segments":

> ... la proportion, qui est entre les droites et les courbes, nest pas connue, et mesme ie croy ne le pouvant pas estre par les hommes, on ne pourroit rien conclure de l qui fust exact et assur.

We have so far, in the spirit of the quote from Mueller, tried to find the proportionality constant only for a specific proportion. In the remainder of the section, we consider several ways of systematizing the solution of families of such problems. First, using the completeness of $\mathcal{E}^2$, we can find a model where every angle determines an arc that corresponds to the length of a straight line segment. Then we consider several model theoretic schemes to organize such problems. Before carrying out the construction we provide some context.

Birkhoff [Bir32] introduced the following axiom in his system[19].

> POSTULATE III. The half-lines ℓ, m, through any point O can be put into $(1,1)$ correspondence with the real numbers $a(\mathrm{mod} 2\pi)$, so that, if $A \neq O$ and $B \neq O$ are points of ℓ and m respectively, the difference $a_m - a_\ell (\mathrm{mod} 2\pi)$ is $\angle AOB$.

This is a parallel to his 'ruler postulate' which assigns each segment a real number length. Thus, Birkhoff takes the real numbers as an unexamined background object. At one swoop he has introduced multiplication, and assumed the Archimedean

[18] Descartes, Oeuvres, vol. 6, p. 412. Crippa also quotes Averros as emphatically denying the possibility of such a ratio and notes Vieta held similar views.

[19] This is the axiom system used in virtually all U.S. high schools since the 1960's.

and completeness axioms. So even 'neutral' geometries studied on this basis are actually are greatly restricted. He argues that his axioms define a categorial system isomorphic to $\Re^2$. So it is equivalent to Hilbert's.

In fact, the separate protractor postulate is not needed. Euclid's 3rd postulate, "describe a circle with given center and radius", implies that a circle is uniquely determined by its radius and center. In contrast Hilbert simply defines the notion of circle and (see Lemma 11.1 of [Har00]) proves the uniqueness. In either case we have: two segments of a circle are congruent if they cut the same central angle. We will now find a more modest version of Birkhoff's postulate: a first order theory with countable models which assign a measure to each angle between 0 and 2π. Recall that we have a field structure on the line through 01 and the number π on that line. We will make one further explicit definition.

Definition 3.8. *A* measurement of angles *function is a map μ from congruence classes of angles into $[0, 2\pi)$ such that if $\angle ABC$ and $\angle CBD$ are disjoint angles sharing the side BC, $\mu(\angle ABD) = \mu(\angle ABC) + \mu(\angle CBD)$*

If we omitted the additivity property this would be trivial: Given an angle $\angle ABC$ less than a straight angle, let C' be the intersection of a perpendicular to AC through B with AC and let $\mu(\angle ABC) = \frac{BC'}{AB}$. (It is easy to extend to the rest of the angles.) To obtain the additivity, we proved in [Bal14b]:

Theorem 3.9. *For every countable model M of $\mathcal{E}^2_\pi$, there is a countable model M' containing M such that a measure of angles function μ is defined on the (congruence class of) each angle determined by points $P, X, Y \in M'$.*

Corollary 3.10. *If M is a countable recursively saturated model of $\mathcal{E}^2_\pi$ a measure of angles function μ is defined on the (congruence class of) each angle determined by points $P, X, Y \in M$.*

We have constructed a countable model M such that each arc of a circle in M has length measured by a straight line segment in M. There is no Archimedean requirement; adding the Archimdean axiom here would determine a unique number rather than a monad. Note that in any model satisfying the hypotheses of Corollary 3.10, we can carry out elementary right angle trigonometry (angles less than 180^o). Unit circle trigonometry, where periodicity extends the sin function to all of the line violates *o*-minimality. (The zeros of the $\sin$ function are an infinite discrete set.)

Remark 3.11 (Gödel completeness again)**.** It might be objected that such minor changes as adding to $\mathcal{E}$ the name of the constant π, or adding the definable functions C and A undermines the claim in Remark 2.2 that $\mathcal{E}^2$ was descriptively complete for Cartesian geometry. But the data set has changed. We add these new constants and functions because the modern view of 'number' requires them.

Descriptive completeness is *not* always preserved by naming constants even to a Gödel -complete theory. Compare Dicta 3.1.

This section was devoted to first order axiomatizations of geometries that include transcendentals. In Section 4.5, we will address some ways to regain categoricity.

4 And back to Hilbert

We discuss first the role of the Archimedean postulate in the Grundlagen. Then we list objections to second order axioms for 'geometry' and suggest in Section 4.4 that Hilbert is conceiving of geometry in a somewhat broader sense. In a final section, we speculate on broader uses of 'definable mathematics'.

4.1 The role of the Axiom of Archimedes in the Grundlagen

The discussions of the Axiom of Archimedes in the Grundlagen fall into several categories. i) Those, in Sections 9 -12 (from [Hil71]) are metamathematical - concerning the consistency and independence of the axioms. ii) In Section 17, the axiom of Archimedes is used to justify the coordinatization of n-space by n-tuples of real numbers. iii) In Sections 19 and 21, it is shown that the Archimedean axiom is necessary to show equicomplementable (equal content) is the same as equidecomposable (in 2 or more dimensions). These are all metatheoretical results.

Coordinatization is certainly a central geometrical notion. But it does not require the axiom of Archimedes to coordinatize n-space by the field defined on the line in the plane. Hilbert uses the axiom to assign (a binary representation of a real) to each point on the line. That is, to establish a correspondence between object defined in the geometry and an extrinsic notion of real number. Thus, it is not a proof in Hilbert's system. The use of the Archimedean axiom to prove equidecomposable is the same as equicomplementable is certainly a proof in the system. But an unnecessary one. As we argued in Section 4.4 of [Bal14a], Hilbert could just have easily defined 'same area' as 'equicomplementable' (as it is in a natural reading of Euclid).

Thus, we find no theorems in the Grundlagen proved from its axiom system that essentially depend on the Axiom of Archimedes. Rather Hilbert's use of the axiom of Archimedes is to create counterexamples to show independence and, in conjunction with the Dedekind axiom, identify the field defined in the geometry with the independent existence of the real numbers as conceived by Dedekind.

4.2 Hilbert and Dedekind on Continuity

Hilbert's formulation of the completeness axiom reads [Hil71]:

> Axiom of Completeness (Vollständigkeit): To a system of points, straight

> lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

We have used in this article the following adaptation of Dedekind's postulate for geometry (DG):

> DG: The linear ordering imposed on any line by the betweenness relation is Dedekind complete.

While this formulation is convenient for our purposes, it misses an essential aspect of Hilbert's version. DG implies the Archimedean axiom and Hilbert was aiming for an independent set of axioms. Hilbert's axiom does not imply Archimedes. A variant VER (see [Can99]) on Dedekind's postulate that does not imply the Archimedean axiom was proposed by Veronese in [Ver89][20]. If we substituted VER for DG, our axioms would also satisfy the independence criterion.

Hilbert's completeness axiom in [Hil71] asserting any model of the rest of the theory is maximal, is inherently model-theoretic. The later line-completeness [Hil62] is a technical variant[21]. Giovannini's account [Gio13] includes a number of points already made here; but I note three further ones. First, Hilbert's completeness axiom is not about deductive completeness (despite having such consequences), but about maximality of every *model* (page 145). Secondly (last line of 153) Hilbert expressly rejects Cantor's intersection of closed intervals axiom because in relies on a sequence of intervals and 'sequence is not a geometrical notion'. A third intriguing note is an argument due to Baldus in 1928 that the parallel axiom is an essential ingredient in the categoricity of Hilbert's axioms [22].

Here are two reasons for choosing Dedekind's (or Veronese's) version. The most basic is that one cannot formulate Hilbert's version as sentence Φ in second order

[20] The axiom VER asserts that for a partition of a linearly ordered field into two intervals L, U (with no maximum in the lower L or minimum in the upper U) and third set in between at most one point, there is a point between L and U just if for every $e > 0$, there are $a \in A, b \in B$ such that $b - a < e$. Veronese derives Dedekind's postulate from his plus Archimedes in [Ver89] and the independence in [Ver91]. In [LC93] Levi-Civita shows there is a non-Archimean ordered field that is Cauchy complete. I thank Philip Ehrlich for the references and recommend section 12 of the comprehensive [Ehr06]. See also the insightful reviews [Pam14b] and [Pam14a] where it is observed that Vahlen [Vah07] also proved this axiom does not imply Archimedes.

[21] Since any point in the definable closure of any line and any one point not one the line, one can't extend any line without extending the model. Since adding either the Dedekind postulate and or Hilbert completeness gives a categorical theory satisfied by a geometry whose line is order isomorphic to $\Re$ the two axioms are equivalent (over HP5 + Arch).

[22] Hartshorne (sections 40-43 of [Har00] gives a modern account of Hilbert's argument that replacing the parallel postulate by the axiom of limiting parallels gives a geometry that is determined by the underlying (definable) field. With V.2 this gives categoricity.

logic[23] with the intended interpretation

$$(\Re^2, \mathbf{G}) \models \Phi. \;\; (*)$$

The axiom requires quantification over subsets of an extension of the model which putatively satisfies it. Here is a second order statement[24] of the axiom, where ψ denotes the conjunction of Hilbert's first four axiom groups and the axiom of Archimedes.

$$(\forall X)(\forall Y)\forall \mathbf{R})[[X \subseteq Y \wedge (X, \mathbf{R}\restriction X) \models \psi \wedge (Y, \mathbf{R}) \models \psi] \rightarrow X = Y]$$

This anomaly has been investigated by Väänänen who makes the distinction between the last two displayed formulas (on page 94 of [Vää12a]) and expounds in [Vää12b] a new notion, 'Sort Logic', which provides a logic with a sentence Φ which by allowing a sort for an extension of the given model axiomatizes geometry in the sense of (*). The second reason is that Dedekind's formulation, since it is about the geometry, not about its axiomatization, directly gives the kind of information about the existence of transcendental numbers that we discuss in the paper.

In [Vää12a], Väänänen discusses the categoricity of natural structures such as real geometry when axiomatized in second order logic (e.g. DG). He has discovered the striking phenomena of 'internal categoricity'. Suppose the second order categoricity of a structure A is formalized by the existence of sentence Ψ_A such that $A \models \Psi_A$ and any two models of Ψ are isomorphic. If this second clause in provable in a standard deductive system for second order logic, then it is valid in the Henkin semantics, not just the full semantics. This observation[25] makes a variant of the argument of point 5 in Section 4.3 apply to those sentences in second order logic that are internally categorical. The earlier argument was to make 2nd (and higher) order arguments to provide semantic proofs that certain first order propositions are derivable in first order logic. Now, one can deduce certain second order propositions can be derived from the formal system of second order logic by employing 3rd (and higher) order arguments to provide semantic proofs.

Philip Ehrlich has made several important discoveries concerning the connections between the two 'continuity axioms' in Hilbert and develops the role of maximality. First, he observes (page 172) of [Ehr95] that Hilbert had already pointed out that his completeness axiom would be inconsistent if the maximality were only with respect to the first order axioms. Secondly, he [Ehr95, Ehr97] systematizes and investigates the philosophical significance of Hahn's notion of Archimedean completeness. Here the structure (ordered group or field) is not required to be Archimedean; the maximality condition requires that there is extension which fails to extend an Archimedean

[23]This analysis is anachronistic; the clear distinction between first and second order logic did not exist in 1900. By $\mathbf{G}$, we mean the natural interpretation in $\Re^2$ of the predicates of geometry in the vocabulary τ.

[24]I am leaving out many details, $\mathbf{R}$ is a sequence of relations giving the vocabulary of geometry and the sentence 'says' they are relations on Y; the coding of the satisfaction predicate is suppressed.

[25]Discussion with Väänänen.

equivalence class[26]. This notion provides a tool (not yet explored) for investigating the non-Archimedean models studied in Section 3. The use in that section of various weakenings of saturation to study these models has natural links with Ehrlich's rephrasing the maximality condition in terms of homogeneous universal structure [Ehr92].

4.3 Against the Dedekind Postulate for Geometry

Our fundamental claim is that (slight variants on) Hilbert's first order axioms provide a modest descriptively complete axiomatization of most of Greek geometry.

As we pointed out in Section 3 of [Bal14c] various authors have proved under $V = L$, any countable or Borel structure can be given a categorical axiomatization. We argued there that this fact undermines the notion of categoricty as an independent desiderata for an axiom system. There, we gave a special role to attempting to axiomatize canonical systems. Here we go further, and suggest that even for a canonical structure there are advantages to a first order axiomatization that trump the loss of categoricity.

We argue then that the Dedekind postulate is inappropriate (in particular immodest) as an attempt to axiomatize the Euclidean or Cartesian or Archimedean data sets for several reasons:

1. The requirement that there be a straight-line segment measuring any circular arc is clearly contrary to the intent of Descartes and Euclid.
2. Dedekind's postlate is not part of the data set but rather an external limitative principle.

   The notion that there was 'one' geometry (i.e. categoricity) was implicit in Euclid. But it is not a geometrical statement. Indeed, Hilbert described his completeness axiom (page 23 of [Hil62]), 'not of a purely geometrical nature'. This is most clearly seen in Hilbert's initial metamathematical formulation: that the model of Axiom groups I-IV and Archimedes axiom must be maximal.
3. It is not needed to establish the properly geometrical propositions in the data set. We spelt out in [Bal14a] the description of two data sets of Euclidean geometry and appropriate sets of first order axioms that provide modest descriptively complete axiomatization for each them. In this paper we argued that Tarski's axioms are a modest descriptive axiomatization of Descartes' geometry and showed a slight extension of Tarski's first order axiomatization accounts not only for the Cartesian data set but the basic properties of π.
4. Proofs from Dedekind's postulate obscure the true geometric reason for certain theorems. Hartshorne writes[27]:

[26] In an ordered group, a and b are *Archimedes-equivalent* if there are natural numbers m, n such that $m|a| > |b|$ and $n|b| > |a|$.

[27] page 177 of [Har00]

'... there are two reasons to avoid using Dedekind's axiom. First, it belongs to the modern development of the real number systems and notions of continuity, which is not in the spirit of Euclid's geometry. Second, it is too strong. By essentially introducing the real numbers into our geometry, it masks many of the more subtle distinctions and obscures questions such as constructibility that we will discuss in Chapter 6. So we include the axiom only to acknowledge that it is there, but with no intention of using it.

5. The use of second order logic undermines a key proof method – informal (semantic) proof. A crucial advantage of a first order axiomatization[28] is that it licenses the kind of argument[29] described in Hilbert and Ackerman[30]:

> Derivation of Consequences from Given Premises; Relation to Universally Valid Formulas
>
> So far we have used the predicate calculus only for deducing valid formulas. The premises of our deductions, viz Axioms a) through f), were themselves of a purely logical nature. Now we shall illustrate by a few examples the general methods of formal derivation in the predicate calculus .... It is now a question of deriving the consequences from any premises whatsoever, no longer of a purely logical nature.
>
> The method explained in this section of formal derivation from premises which are not universally valid logical formulas has its main application in the setting up of the primitive sentences or axioms for any particular field of knowledge and the derivation of the remaining theorems from them as consequences . We will examine, at the end of this section, the question of whether every statement which would intuitively be regarded as a consequence of the axioms can be obtained from them by means of the formal method of derivation.

We exploited this technique in Sections 3 to calculate the circumference and area of a circle.

Venturi[31] formulates a distinction, which nicely summarizes our argument: 'So we can distinguish two different kinds of axioms: the ones that are *necessary* for the development of a theory and the *sufficient* one used to match intuition and formalization.' In our terminology only the necessary axioms make up a '*modest* descriptive axiomatization'. For the geometry Euclid I (basic polygonal geometry), Hilbert's first order axioms meet this goal. With $\mathcal{E}^2_\pi$, a modest complete descriptive axiomatization is provided even including the basic properties of π. The Archimedes and Dedekind

[28] See Section 4.2 for more detail on this argument.

[29] We noted that Hilbert proved that a Desarguesian plane in 3 space by this sort of argument in Section 2.4 of [Bal13].

[30] Chapter 3, §11 Translation taken from [Bla14].

[31] page 96 of [Ven11]

postulates have a different goal; they secure the 19th century conception of $\Re^2$ to be the unique model and thus ground elementary analysis.

4.4 Why does axiom group V exist?

Hilbert wrote that his axioms V.1 and V.2 allow one 'to establish a one-one correspondence between the points of a segment and the system of real numbers'. Here he is exhibiting a model centric rather than a descriptive (in Detlefsen's syntactic sense) approach to axiomatization. I think these should be recognized as two distinct motivations: to make Euclid rigorous and to ground analytic geometry and calculus. These are complementary but distinct projects. We have noted here that the grounding of real algebraic geometry is fully accomplished by Tarski's axiomatization. And we have provided a first order extension to deal with the basic properties of the circle.

Hilbert has in fact a more modern notion of geometry which includes both 'metric' and 'algebraic' geometry. Both metric and algebraic geometers regard themselves as geometers while recognizing the differences in goals and methods among the subjects. Hilbert seems really to be aiming at the foundations of both. But his devotion of the text to axiomatic treatment of Euclidean geometry suggests a more narrow reading of his goals.

Perhaps the confusion arises from two of the many meanings of 'complete' in modern mathematics. Particularly because his logical formulation, one is led to think his completeness axiom is about semantic completeness. But careful analysis of his writings (See [Gio, Ven11].) shows in trying to ground analytic geometry he was really focused on order completeness.

4.5 Regaining categoricity?

We noted in Section 2 of [Bal14a] that categoricity was a goal of the American postulate theorists. Categoricity per se can be conflated with a belief that Dedekind indeed had described 'the continuum' and the goal was to specify exactly that object. We briefly describe a number of geometries which have a unique model that is countable.

The quote from Mueller [Mue06] in Section 3, inspires a search for a principle to distinguish the application of the method of Eudoxus by the Greeks to solve particular problems from the *Deus ex machina* of Dedekind: fill all cuts. Putatively, to state a problem is to state it recursively, so an answer would be to fill the recursive cuts.

The logic $L_{\omega_1,\omega}$ allows us to describe several possibilities. The sentence that adds to RCF the axiom of Archimedes plus the assertion that exactly the recursive cuts over the surd field are realized has exactly one model. This could be varied by fixing a smaller subfield of the real algebraic numbers as a base or by realizing a different countable set of cuts. Since the Archimedean principle implies each cut has at most one realization each of these sentences has a unique model. In general, they will not satisfy the angle measure principle. But restricting the construction for Theorem 3.9 to Archimedean fields the property could be guaranteed. More expansively, the Scott sentence[32] of the recursively saturated model would give a richer theory with a unique countable model (but also uncountable models).

These examples illustrate that finding an axiom set with a unique model becomes interesting when the model itself is mathematically important. In contrast as we discussed at length in [Bal14c], first order categoricity in an uncountable power has powerful structural consequences for any model.

4.6 But what about analysis?

We have expounded a procedure [Har00] to define the field operations in an arbitrary Euclidean plane. We argued that the first order axioms of EG suffice for the geometrical data sets Euclid I and II, not only in their original formulation but by finding proportionality constants for the area formulas of polygon geometry. By adding axioms to require the field is real closed we obtain a complete first order theory that encompasses many of Descartes innovations. The plane over the real algebraic numbers satisfies this theory; thus, there is no guarantee that there is a line segment of length π. Using the o-minimality of real closed fields, we can guarantee there is such a segment by adding a constant for π and requiring it to realize the proper cut in the rationals. However, guaranteeing the uniqueness of such a realization requires the $L_{\omega_1,\omega}$ Archimedean axiom.

Hilbert and the other axiomatizers of 100 years ago wanted more; they wanted to secure the foundations of calculus. In full generality, this surely depends on second order properties. But there are a number of directions of work on 'definable analysis'.

One of the directions of research in o-minimality has been to prove the expansion of the real numbers by a particular functions (e.g. the Γ-function on the positive reals [SvdD00]); our example with arc length might be extended to consider other more interesting transcendental curves.

Peterzil and Starchenko study the foundations of calculus in [PS00]. They approach the complex analysis through o-minimality of the real part in [PS10]. The impact of o-minimality on number theory was recognized by the Karp prize of 2014. And a non-logician, suggests using methods of Descartes to teach Calculus [Ran14]. For an interesting perspective on the historical background of the banishment of infinitesimals in analysis see [BK12].

In a sense, our development is the opposite of Ehrlich's in [Ehr12]. Rather than trying to unify all numbers great and small, we are interested in the minimal

[32] In general the Scott sentence of a countable model [Kei71] is a sentence of $L_{\omega_1,\omega}$ that specifies the model up to isomorphism among countable models.

collection of numbers that allow the development of a geometry according with our fundamental intuitions.

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