Laurent Series

**Theorem.** Let $f(z)$ be analytic in the closed region

$$D_{R_1,R_2} = \{0 < R_1 \leq |z| \leq R_2\}.$$

Then for $R_1 < |z| < R_2,$

$$f(z) = \frac{1}{2\pi i} \oint_{C_{R_2}} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{C_{R_1}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

**Proof:** The proof is similar in spirit to the proof of the Cauchy Integral Formula. Fix $z.$ For $\epsilon$ small, let $C_{z,\epsilon} = \{\zeta ||\zeta - z| = \epsilon\}$ is in between $C_{R_1}$ and $C_{R_2}.$ Define $g_z(\zeta) = \frac{f(\zeta)}{\zeta - z}.$ Then $g_z(\zeta)$ is an analytic function of $\zeta$ in the region between the two circles $C_{R_1}$ and $C_{R_2}$ and outside $C_{z,\epsilon}.$

\[
2\pi i \ f(z) = \oint_{C_{z,\epsilon}} g_z(\zeta) \, d\zeta \\
= \oint_{C_{R_2}} g_z(\zeta) \, d\zeta - \oint_{C_{R_1}} g_z(\zeta) \, d\zeta \\
= \oint_{C_{R_2}} \frac{f(\zeta)}{z - \zeta} \, d\zeta - \oint_{C_{R_1}} \frac{f(\zeta)}{z - \zeta} \, d\zeta.
\]

Next we write

\[
f(z) = \frac{1}{2\pi i} \oint_{C_{R_2}} \frac{f(\zeta)}{z - \zeta} \, d\zeta - \frac{1}{2\pi i} \oint_{C_{R_1}} \frac{f(\zeta)}{z - \zeta} \, d\zeta \\
= f_1(z) + f_2(z).
\]
Proceeding as before

\[ f_1(z) = \frac{1}{2\pi i} \oint_{C_{R_2}} \frac{f(\zeta)}{\zeta - z} \, d\zeta. \]

\[ = \frac{1}{2\pi i} \oint_{C_{R_2}} \frac{f(\zeta)}{\zeta} \left( \sum_{n=0}^{\infty} \left( \frac{z}{\zeta} \right)^n \right) \, d\zeta \]

\[ = \frac{1}{2\pi i} \oint_{C_{R_2}} \frac{f(\zeta)}{\zeta} \left( \sum_{n=0}^{N} \left( \frac{z}{\zeta} \right)^n \right) \, d\zeta \]

\[ + \frac{1}{2\pi i} \oint_{C_{R_2}} \frac{f(\zeta)}{\zeta} \left( \frac{z}{\zeta} \right)^{N+1} \, d\zeta \]

\[ = \sum_{n=0}^{\infty} a_n z^n, \]

where

\[ a_n = \frac{1}{2\pi i} \oint_{C_{R_2}} f(\zeta) \frac{1}{\zeta^{n+1}} \, d\zeta. \]

Note that \( f_1(z) \) is analytic in the region \( \{z \mid |z| \leq R_2\} \).

In the same spirit,

\[ f_2(z) = -\frac{1}{2\pi i} \oint_{C_{R_1}} \frac{f(\zeta)}{\zeta - z} \, d\zeta \]

\[ = \frac{1}{2\pi i} \oint_{C_{R_1}} \frac{f(\zeta)}{z} \frac{1}{1 - \frac{\zeta}{z}} \, d\zeta \]

\[ = \frac{1}{2\pi i} \oint_{C_{R_1}} \frac{f(\zeta)}{z} \left( \sum_{m=0}^{\infty} \left( \frac{\zeta}{z} \right)^m \right) \, d\zeta \]

\[ = \frac{1}{2\pi i} \oint_{C_{R_1}} \frac{f(\zeta)}{z} \left( \sum_{m=0}^{M} \left( \frac{\zeta}{z} \right)^m \right) \, d\zeta \]

\[ + \frac{1}{2\pi i} \oint_{C_{R_1}} \frac{f(\zeta)}{z} \left( \frac{\zeta}{z} \right)^{M+1} \, d\zeta \]

\[ = \sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \left( \frac{1}{2\pi i} \oint_{C_{R_2}} f(\zeta) \zeta^m \, d\zeta \right). \]

Note that \( f_2(z) \) is analytic in the region \( \{z \mid |z| \geq R_1\} \).

The Laurent Expansion
Theorem. Let $f(z)$ be analytic in the region $\{ z | R_1 < |z| < R_2 \}$. Then for $R_1 < |z| < R_2$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

$$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta)\zeta^{-n-1} d\zeta.$$

Here $r$ is any number such that $R_1 < r < R_2$.

The series

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in $\{ z | |z| < R_2 \}$.

The series

$$f_2(z) = \sum_{n=-\infty}^{-1} a_n z^n$$

is analytic in $\{ z | R_1 < |z| \}$.

Consequences and Notes

- If $f(z)$ be analytic in the region $\{ z | |z| < R_2 \}$, then

$$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta)\zeta^{-n-1} d\zeta = 0, n = -1, -2, \ldots.$$  
  
- If $f(z)$ be analytic in the region $\{ z | 0 < |z| < R_2 \}$, then

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) d\zeta$$

is called the residue of $f(z)$ at $z = 0$.

Zeroes, Poles, and Essential Singularities

For the moment, we shall consider a function $f(z)$ analytic in the punctured disk

$$\hat{D}_R = \{ z | 0 < |z| \leq R \}.$$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

$$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta)\zeta^{-n-1} d\zeta.$$

- If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(z)$ may be extended by defining $f(0) = a_0$, and the resulting function is analytic in $|z| \leq R$.  

• If \( f(z) = \sum_{n=N}^{\infty} a_n z^n, \ N \geq 0, \ a_N \neq 0, \ f(z) \) is said to have a zero of order \( N \) at \( z = 0 \).
Near \( z = 0, \quad f(z) = z^N \cdot g(z) \)
, where \( g(z) \) is analytic in \(|z| \leq R, \ g(0) \neq 0\).

• If \( f(z) = \sum_{n=-M}^{\infty} a_n z^n, \ M \geq 0, \ a_{-M} \neq 0, \ f(z) \) is said to have a pole of order \( M \) at \( z = 0 \).
Near \( z = 0, \quad f(z) = z^{-M} \cdot g(z) \)
, where \( g(z) \) is analytic in \(|z| \leq R, \ g(0) \neq 0\).

• If \( f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \ a_n \neq 0 \) for infinitely many negative \( n \), then \( f(z) \) is said to have an essential singularity at \( z = 0 \).

• The coefficient of \( z^{-1} \) is called the residue of \( f(z) \) at \( z = 0 \), and is written
\[
\text{Res}(f, z = 0) = \text{Res} f(z)|_{z=0} = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) \, d\zeta.
\]

Exercises
1. Let \( f(z) \) be analytic in the punctured disk
\[ \dot{D}_R = \{ z \mid 0 < |z| \leq R \} \]
Then for \( r \) small and positive,
\[ \oint_{C_r} f(\zeta) \, d\zeta = 2\pi i \text{Res} f(z)|_{z=0}. \]

2. Let \( f(z) \) be analytic in the punctured disk
\[ \dot{D}_R = \{ z \mid 0 < |z| \leq R \} \]
Suppose that \( f(z) \) has a zero of order \( N > 0 \), at \( z = 0 \).
Then for \( r \) small and positive,
\[ \oint_{C_r} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = 2\pi i \cdot N. \]

3. Let \( f(z) \) be analytic in the punctured disk
\[ \dot{D}_R = \{ z \mid 0 < |z| \leq R \} \]
Suppose that \( f(z) \) has a pole of order \( M > 0 \), at \( z = 0 \).
Then for \( r \) small and positive,
\[ \oint_{C_r} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = -2\pi i \cdot M. \]

4. Let \( f(z) \) be analytic in the punctured disk
\[ \dot{D}_R = \{ z \mid 0 < |z| \leq R \} \]
Suppose that \( f(z) \) is bounded as \( z \to 0 \). Show that
• \( \lim_{z \to 0} f(z) \) exists.
• \( f(z) \) may be extended to be an analytic function in
\[ D_R = \{ z \mid |z| \leq R \} \]
As a consequence, the singularity of \( f(z) \) at \( z = 0 \) is removable.
5. Let $f(z)$ be analytic in the punctured disk

$$\hat{D}_R = \{z | 0 < |z| \leq R \}.$$ 

Suppose that

$$f(z) = O\left(|z|^M\right)$$
as $z \to 0$.

Show that for $n < M$, $a_n = 0$. 
