Required Part:

0. Read §1f Extrema of Quadratic Forms, §3a Univariate Models, and §3b Sampling Distributions.

1. Let $\Sigma$ be an $n \times n$ p.d. matrix. The inner product of $\mathbb{R}^n$ is defined by $(x, y) = x'\Sigma y$. Show that given an arbitrary $n \times n$ matrix $A$, an orthogonal projector onto $\mathcal{M}(A)$ is

$$P = A(A'\Sigma A) - A'A\Sigma.$$  

Hint: See (vi) on page 47.

2. Let $A$ be an $m \times m$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and corresponding orthonormal eigenvectors $P_1, \ldots, P_m$. Consider $k$ ($k \leq m$) mutually orthogonal vectors $x_1, \ldots, x_k$ in $\mathbb{R}^m$. Show that

$$\sup_{x_1, \ldots, x_k} \sum_{i=1}^{k} \frac{x_i'Ax_i}{x_i'x_i} = \sum_{i=1}^{k} \lambda_i,$$

where the supremum is attained at $x_i = cP_i$, $i = 1, \ldots, k$ for some scalar $c \neq 0$.

Hint: See (iv) on page 63.

3. Let $A$ be an $n \times n$ symmetric matrix and $B$ be an $m \times n$ matrix. Suppose a random vector $Y \sim N_n(\mu, \sigma^2 I_n)$, where $\mu = (\mu_1, \ldots, \mu_n)'$. Show that $Y'AY$ and $BY$ are independent if $BA = 0$.

Hint: See Problem 1.2 on page 209.

4. Suppose $Y \sim N_n(X\beta, \sigma^2 I_n)$, where $Y = (y_1, \ldots, y_n)'$ is a random vector, $X$ is an $n \times m$ matrix with rank $r$, and $\beta = (\beta_1, \ldots, \beta_m)'$ is a vector of parameters. Let $\hat{\beta} = (X'X)^{-1}X'Y$ which is a solution to $X'X\beta = X'Y$. Let $H$ be an $m \times k$ matrix, such that, $\mathcal{M}(H) = \mathcal{M}(X')$. Denote $Z = H'\hat{\beta}$ and $R_0^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta})$.

(a) Show that $X = X(X'X)^{-1}X'X$.

(b) Show that $I - X(X'X)^{-1}X'$ is idempotent.

(c) Show that $Z$ and $R_0^2$ are independent.

(d) Show that $Z \sim N_k(H'\beta, \sigma^2 H'(X'X)^{-1}H)$, and $R_0^2 \sim \sigma^2 \chi^2(n - r)$.

Hint: See Problems 2.1, 2.2, and 2.3 on page 210.

Optional Part:

5. Let $A$ and $B$ be $n \times n$ symmetric matrices. Suppose a random vector $Y \sim N_n(\mu, \sigma^2 I_n)$, where $\mu = (\mu_1, \ldots, \mu_n)'$. Show that $Y'AY$ and $Y'BY$ are independent if $AB = 0$ or $BA = 0$.

Hint: See Problem 1.1.1 on page 209.