Required Part:

0. Read §8b Wishart Distribution and §4a Theory of Least Squares (Linear Estimation).

For Problems 1, 2, and 4, we consider a linear model \( Y = X\beta + \varepsilon \), where \( Y \) is an \( n \times 1 \) random vector, \( X \) is a known \( n \times m \) matrix, \( \beta \) is an \( m \times 1 \) vector of unknown parameters, and \( \varepsilon \) is an \( n \times 1 \) random vector of noise with \( E(\varepsilon) = 0 \) and \( V(\varepsilon) = \sigma^2 I_n \). All \( L, C, D, P, L_0 \) are column vectors.

1. Recall that a linear function \( P'\beta \) of \( \beta \) is called **estimable** if there exists a linear function \( L'Y \) such that \( E(L'Y) = P'\beta \).

   (a) Based on the definition of estimable linear function, show that all estimable linear functions of \( \beta \) form a linear space, that is,
   * if \( C'\beta \) and \( D'\beta \) are estimable, then \( (C + D)'\beta \) is estimable too;
   * if \( C'\beta \) is estimable and \( a \) is a scalar, then \( (aC)'\beta \) is estimable.

   (b) Determine the linear space formed by all estimable linear functions of \( \beta \).

2. Consider equation \( P = X'L \) where both \( P \) and \( L \) are \( n \times 1 \) vectors.

   (a) Show that if \( P = X'L \) admits a solution for \( L \), then \( L'Y \) is unbiased for \( P'\beta \).

   (b) Suppose \( P = X'L \) admits a solution for \( L \). Show that there exists a unique solution \( L = L_0 \in \mathcal{M}(X) \).

   (c) Show that \( V(L_0'Y) \leq V(L'Y) \) for any other solution \( L \).

3. Consider the linear model \( y_{ij} = \beta_0 + \beta_i + \varepsilon_{ij}, \quad i = 1, 2, 3, \quad j = 1, 2 \) with the standard assumptions on \( \varepsilon \), that is, \( \varepsilon_{ij} \) iid \( \sim N(0, \sigma^2) \).

   (a) Show that \( c_1\beta_1 + c_2\beta_2 + c_3\beta_3 \) is estimable if and only if \( c_1 + c_2 + c_3 = 0 \).

   (b) Suppose the observations are \( y_{11} = 0.1, \quad y_{12} = -0.5, \quad y_{21} = 3.2, \quad y_{22} = 6.3, \quad y_{31} = 4.9, \) and \( y_{32} = 5.9 \). Find the normal equations.

   (c) For the data in (b), find BLUEs for \( \beta_1 - \beta_2, \beta_1 - \beta_3, \) and \( \beta_2 - \beta_3 \).

Optional Part:

4. Show that in general a linear function \( L'Y \) has minimum variance as an linear unbiased estimate of \( E(L'Y) \) if and only if \( cov(L'Y, D'Y) = 0 \) for all \( D \) such that \( E(D'Y) = 0 \).