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Invariants of theta-curves and other graphs in 3-space*

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Abstract

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Given a graph in 3-space, in general knotted, can one construct a surface containing the graph in some canonical way so that the embedding of the surface in space, or even the link type of its boundary, is an invariant of the knotted graph? We consider, in particular, surfaces that contain the graph as a spine and that are canonical in the sense of having trivial Seifert linking form. It turns out that θ-curves and K_4-graphs are the unique graphs for which this approach works.

Keywords: Knotted graphs, knot theory.

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We are interested in topological invariants of graphs embedded in 3-dimensional space. A primary method of constructing such invariants is to associate a collection of knots and links to a graph such that the isotopy class of the collection is an invariant of the isotopy type of the graph (for example, see [5, 6, 13]). In this paper,

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we obtain a new invariant by associating a link to a graph with 3-valent vertices (called a trivalent graph). The paper is organized as follows:

In Section 2, we discuss the possibility of associating a link with a trivalent graph, which is the boundary of an orientable surface of zero Seifert form collapsing to the graph, as a knot type invariant of the graph. We prove that the θ-curves and the K_4-graphs are the only 3-valent graphs to which such an invariant link can be associated. The proof also provides a method of constructing the invariant link. In Section 3 we show that the θ-curve spines are uniquely determined by the surfaces while the similar statement is not true for the K_4-graphs. Thus there is hope that those links can completely classify knotted θ-curves, while this cannot work for K_4-graphs. As affirmative examples and applications, the invariant links can distinguish θ-curves through five crossings and K_4-graphs through four crossings (as listed in the tables in [11]). In Section 4 we remark that for graphs with vertices of valence greater than 3 it is impossible to define the invariant link in this manner.

1. Preliminaries and notations

Let $\mathbb{R}^3$ be the 3-dimensional Euclidian space, and $\mathbb{S}^3$ be the 3-sphere. We shall work in piecewise-linear category (but we shall draw figures piecewise-smooth just for convenience), and work on graphs in $\mathbb{R}^3$. Every result here is true for graphs in $\mathbb{S}^3$.

A *knotted graph*, or an *embedded graph*, or simply a *graph* Γ in $\mathbb{R}^3$ (or $\mathbb{S}^3$) is a one-dimensional polyhedron. Each $x \in \Gamma$ has a neighborhood $U(x) \subseteq \Gamma$ consisting of a finite number of half-open arcs ending at x. The *valence* of a point $x \in \Gamma$ is the number of such arcs (which is independent of the choice of such neighborhood). We only consider one-dimensional polyhedra for which every point has valence $\geqslant 2$, and define *vertex* to be any point in Γ of valence $\geqslant 3$. A polygonal line connecting two vertices (either distinct or identical) and having no vertices in its interior is called an *edge*.

An *abstract graph* is a homomorphism class of knotted graphs and is denoted G. Often, $\Gamma(G)$ is used to denote a knotted graph which belongs to the homeomorphism class G, and is called an embedding of G into $\mathbb{R}^3$ (or $\mathbb{S}^3$).

Two graphs are said to be *equivalent* if they are ambient isotopic ($\cong$). The equivalence class of a graph Γ is called the *knot type* of Γ.

A graph Γ is *planar* if it is equivalent to some graph Γ_0 in $\mathbb{R}^2$ (or $\mathbb{S}^2$).

An abstract graph G is *planar* if there is a planar embedding $\Gamma(G)$.

An abstract graph G is *connected* if some $\Gamma(G)$ is connected as subspace of $\mathbb{R}^3$ (or $\mathbb{S}^3$).

Let Γ be a knotted graph in $\mathbb{R}^3$, P be any plane in $\mathbb{R}^3$ and $p:\mathbb{R}^3 \to P$ the orthogonal projection. The projection p is *regular* for Γ provided p is locally one-one and for each $y \in p(\Gamma), p^{-1}(y)$ contains either one or two points; if $p^{-1}(y)$ contains two points x_1, x_2, then neither of them is a vertex of Γ; and x_1, x_2 have neighborhoods

$A_1, A_2 \subset \Gamma$ that are open arcs such that the projected arcs $p(A_1)$ and $p(A_2)$ meet transversely. In Fig. 1, we illustrate a regular projection of a "theta-curve" (θ-curve), a graph consisting of two vertices joined by three edges. This example shows that a graph may be nontrivially knotted while each of the knots and links in it is trivial [7].

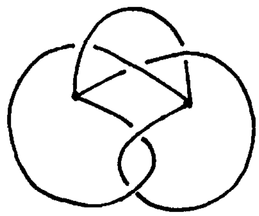


Fig. 1. A regular projection of Kinoshita's θ-curve.

Let S be an oriented surface, x and y be cycles on S. Let x^+ denote the result of pushing x a very small amount into $\mathbb{S}^3 - S$ (or $\mathbb{R}^3 - S$) along the positive normal direction to S. The function $\langle , \rangle : H_1(S) \times H_1(S) \to \mathbb{Z}$ defined by

$$\langle x, y \rangle = \operatorname{lk}(x^+, y)$$

is called the *Seifert form* (or *Seifert linking form*) for S; $\langle x, y \rangle$ is called the *Seifert pairing* of x and y. It is a well-defined, bilinear pairing, an invariant of the ambient isotopy class of the embedding $S \subset \mathbb{S}^3$ (or $\mathbb{R}^3$). Let $x \cdot y$ be the intersection number of x and y; then (well known from Seifert; see e.g. [4])

$$\langle x, y \rangle - \langle y, x \rangle = x \cdot y.$$

We need to consider several ways of decomposing a knotted graph into "simpler" graphs.

Let Γ be a knotted graph. We say Γ is *splittable* if there exists a 2-sphere S in $\mathbb{S}^3 - \Gamma$ which splits $\mathbb{S}^3$ into 3-balls A and B with both $A \cap \Gamma$ and $B \cap \Gamma$ nonempty. The 2-sphere S is called a *splitting sphere.*

Following [12], we say Γ is a *connected sum along a point* (or a *connected sum of type* I) if there exists a 2-sphere S, called an *admissible sphere of type* I, which meets Γ in a vertex w, or transversally in one point w in the interior of an edge and splits $\mathbb{S}^3$ into 3-balls A and B with $(\Gamma - w) \cap A$ and $(\Gamma - w) \cap B$ nonempty. The graph Γ is a *connected sum along two points* (or a *connected sum of type* II) if there exists a 2-sphere S, called an *admissible sphere of type* II, which meets Γ in two points, u and v (each point u, v is either a vertex or is a point of transversal intersection of S with an open edge of Γ), and splits $\mathbb{S}^3$ into 3-balls A and B with neither $(\Gamma - u \cup v) \cap A$ nor $(\Gamma - u \cup v) \cap B$ empty or equal to an unknotted arc, and such that the annulus obtained by taking away small open regular neighborhoods of u and v from S is incompressible in $\mathbb{S}^3 - \Gamma$.

We say Γ is *prime* if (a) it is nonsplittable and (b) it cannot be decomposed into a connected sum of type I or type II (in [12], all planar embeddings are considered nonprime; we do not make this extra distinction).

The above definitions can be converted into similar definitions for $\Gamma \subseteq \mathbb{R}^3$. In the case where $\Gamma \subset \mathbb{R}^2$, the 2-sphere S intersects the plane transversally in a circle C, which we call an admissible circle of type I or II.

2. Association of a link to a knotted graph

2.1. Association of a family of surfaces to a knotted graph

To investigate the possibility of associating a link to a knotted graph as a knot type invariant, let us look at the abstract graphs first. The interest here is the connected graphs. We pass from a graph to a surface closely related to the graph and then take the boundary of that surface.

Lemma 2.1. *For any abstract graph G and any $\Gamma(G)$, there is an orientable surface $S(\Gamma)$ containing $\Gamma(G)$ and collapsing to $\Gamma(G)$.*

Proof. The lemma can be proved by constructing such a surface. Take a regular projection of Γ and isotope Γ such that near each vertex, all edges lie in a small disk parallel to the projection plane P. Put such a disk at each vertex; then connect disks with bands, one along each edge (see Fig. 2). Do this so that each band projects locally one-one onto its image. Let $S(\Gamma)$ be the disk/band surface constructed as above; then it is an orientable surface collapsing to Γ. $\square$

Henceforth, we shall use $S(\Gamma)$ to denote an orientable surface collapsing to a graph Γ. We note that there are many such surfaces since, for example, it is possible to give full twists to the bands in the above construction.

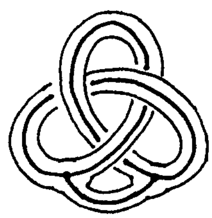


Fig. 2. A disk/band surface.

2.2. Existence and uniqueness of special surfaces

We now begin to consider the questions of existence and uniqueness of surface $S(\Gamma)$ on which the Seifert form is required to be zero. For brevity we call such a surface a *special* (or *good*) surface. We first show that to have a good surface the corresponding abstract graph must be planar.

Lemma 2.2. *If G is a nonplanar abstract graph, then for any embedding $\Gamma(G)$ there are no surfaces of zero Seifert form collapsing to $\Gamma(G)$.*

Proof. If $\Gamma(G)$ is an embedding of a nonplanar abstract graph then $S(\Gamma)$ must have genus $\geqslant 1$, since otherwise $S(\Gamma)$ is homeomorphic to a surface in $\mathbb{R}^2$, which contradicts the hypothesis that G is nonplanar. So there are two cycles, a and b, on S, intersecting each other transversally at one point (Fig. 3). Since $\langle a, b\rangle - \langle b, a\rangle = a \cdot b \neq 0$, the Seifert form of $S(\Gamma)$ is nonzero. □

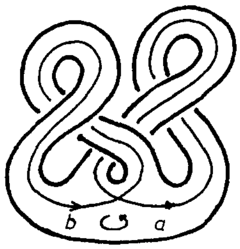


Fig. 3. A nonplanar surface.

Lemma 2.3. *Let G be a connected planar abstract graph; then any two planar embeddings of G are equivalent.*

Proof. Suppose $\Gamma(G)$ and $\Gamma'(G)$ are two embeddings of G in $\mathbb{R}^2$. Then there is a homomorphism ϕ of $\mathbb{R}^3$ to $\mathbb{R}^3$ with $\phi(\Gamma) = \Gamma'$, and ϕ is realizable by an isotopy [8]. □

Theorem 2.4. *Let $\Gamma_0(G)$ be a planar embedding of a connected trivalent planar abstract graph of G. Suppose $\Gamma_0(G)$ is prime.*

(1) *If the number of edges in G is $\leqslant 6$, then for each $\Gamma(G)$, there exists a unique (up to ambient isotopy) surface $S(\Gamma)$ with zero Seifert form.*

(2) *If the number of edges in G is >6 then*
   (i) *there exists a $\Gamma(G)$ with no $S(\Gamma)$ of zero Seifert form;*
   (ii) *if there is an $S(\Gamma)$ of zero Seifert form, it is the unique such surface.*

We state and prove two lemmas for Theorem 2.4.

A graph $\Gamma_0(G)$ has complementary domains $D_1, \ldots, D_n$ that are bounded and one unbounded U. The boundary c_1 of D_1 is a 1-complex which can be viewed as a 1-cycle in $H_1(\Gamma_0)$. We call these *boundary cycles* in Γ_0.

Lemma 2.5. *If G is a connected planar abstract graph, then the set of all boundary cycles in Γ_0 represents a basis of $H_1(\Gamma_0)$.*

Proof. Each loop in Γ_0 can be expressed homologically as a combination of boundary cycles since it is the boundary of a closed region in $\mathbb{R}^2$ which consists of closures of several D_i's, and since filling in the disks bounded by the boundary cycles produces $\mathbb{R}^2 - U$ which, by Alexander duality, has trivial first homology. On the other hand, all boundary cycles are linearly independent: for each boundary cycle, say c_i, let $K_i = \Gamma_0 \cup D_1 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_n$, then it is known that c_i is homologous to 0 in the closure of D_i; if c_i is homologous to 0 in K_i, then $c_i = \partial q_2$ in K_i, so $\partial(q_2 - D_i) = 0$, and $[q_2 - D_i] \in H_2(\mathbb{R}^2 - U)$ is nonzero. But $\mathbb{R}^2 - U$ has trivial second homology, we get a contradiction. Hence c_i is homologically nontrivial in K_i. □

Lemma 2.6. *For a prime planar embedding $\Gamma_0(G)$ of a connected, trivalent graph G with at least four edges,* (1) *every boundary cycle contains at least three edges*; (2) *any two boundary cycles are either disjoint or with one common edge*; (3) *if an edge belongs to more than one boundary cycle, it is a common edge of exactly two boundary cycles*; (4) *every edge belongs to a boundary cycle*; *and* (5) *in each boundary cycle, there is at most one edge which is not a common edge* (*i.e., it belongs to only one boundary cycle, and is called a boundary edge*).

Proof. Use definition of prime and facts in point set topology. We omit the details. □

Proof of Theorem 2.4. Assign some orientation to each edge of the graph Γ; later in the proof we will choose special orientations for certain graphs. Let $S(\Gamma)$ be an orientable surface which collapses to Γ. If ε is sufficiently small, then the boundary of the ε-neighborhood of a vertex intersects Γ transversally at three points. $S(\Gamma)$ can be isotoped near each vertex such that it intersects the ε-ball about each vertex transversally in a disk parallel to a certain plane P which is the plane of a regular projection of Γ. $S(\Gamma)$ may also intersect the ε-neighborhood of each edge transversally in a band which connects the two disks associated to the two vertices of the edge. By the Regular Neighborhood Theorem and Isotopy Extension Theorem [3] there is a space isotopy under which $S(\Gamma)$ is equivalent to a disk/band surface constructed by putting a disk (parallel to the plane of the regular projection of the graph) at each vertex, connecting all disks with bands along the edges as we did in Lemma 2.1, and perhaps adding twists. For such a surface, the Seifert pairing can be easily calculated by counting the corresponding crossing signs of the graph and the number of half twists of bands. So we will only work on disk/band surfaces.

We make the following conventions: the half twist will be counted +1; the half twist will be counted −1; the oriented crossings and will be given signs of +1 and −1, respectively. Let n_i be the number of half twists on the band corresponding to the edge a_i. We also use w_{ij} to denote the sum of the crossing signs of oriented edges a_i and a_j.

We first prove statement (1) of the theorem.

If the number of edges in G is $\leqslant 6$, then being trivalent, G must either consist of two vertices and three edges, or consist of four vertices and six edges. Among these graphs the only ones which have prime planar embeddings must be the θ-curve and the K_4-graph (the complete graph of four vertices).

We claim that for each regular projection of the embedded θ-curve or K_4-graph there is a unique choice of the number of twists on bands to make the corresponding disk/band surface be of zero Seifert form.

In the case of a θ-curve, the three edges can be denoted by a_1, a_2 and a_3 and oriented as in Fig. 4. Let $S(\Gamma)$ be a disk/band surface about Γ. By Lemma 2.5, the generators for $H_1(S(\Gamma))$ can be chosen as

$$\alpha = a_1 - a_2, \qquad \beta = a_2 - a_3.$$

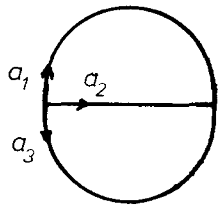


Fig. 4. Oriented and labeled θ-curve.

If $\gamma = \alpha + \beta = a_1 - a_3$, then $\langle \gamma, \gamma \rangle = 0$ if and only if $\langle \alpha, \beta \rangle = 0$. Take a regular projection of Γ; the corresponding disk/band surface is of zero Seifert form if and only if $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 0$, which is

$$n_1 + n_2 = -2(w_{11} + w_{22} - w_{12}),$$

$$n_2 + n_3 = -2(w_{22} + w_{33} - w_{23}),$$

$$n_1 + n_3 = -2(w_{11} + w_{33} - w_{13}).$$

Since the matrix of the equation system with variables n_1, n_2 and n_3 is nonsingular, and the constants are even, there is a unique solution $\{n_1, n_2, n_3\}$. Therefore, there is a unique disk/band surface corresponding to this regular projection of Γ, which is of zero Seifert form.

In the case of a K_4-graph, the six edges can be ordered and oriented as in Fig. 5.

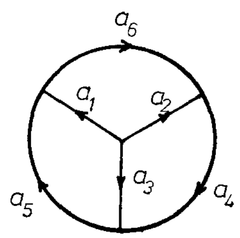


Fig. 5. Oriented and labeled K_4-graph.

Homology generators of $H_1(S(\Gamma))$ can be chosen as

$$\alpha = a_2 - a_3 + a_4,$$

$$\beta = -a_1 + a_3 + a_5,$$

$$\gamma = a_1 - a_2 + a_6.$$

For a regular projection of Γ, the corresponding disk/band surface is of zero Seifert form if and only if

$$n_1 = -w_{23} - w_{25} + w_{21} - 2w_{11} + w_{13} + w_{15} + w_{36} - w_{16} + w_{56},$$

$$n_2 = -w_{24} + w_{14} + w_{46} + w_{23} - w_{13} - w_{36} - 2w_{22} + w_{12} + w_{26},$$

$$n_3 = w_{34} - w_{14} + w_{45} + w_{23} + w_{25} - w_{12} - 2w_{33} + w_{13} - w_{35},$$

$$n_2 + n_3 + n_4 = -2(w_{24} - w_{34} - w_{23} + w_{44} + w_{33} + w_{22}),$$

$$n_1 + n_3 + n_5 = -2(w_{35} - w_{15} - w_{13} + w_{11} + w_{33} + w_{55}),$$

$$n_1 + n_2 + n_6 = -2(-w_{12} + w_{16} - w_{26} + w_{11} + w_{22} + w_{66}).$$

The coefficient matrix of this equation system with unknowns $n_1, \ldots, n_6$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which is nonsingular. Therefore, there is a unique solution $\{n_1, n_2, \ldots, n_6\}$, hence a unique disk/band surface corresponding to the regular projection of Γ.

As we mentioned previously in the proof of Lemma 2.1, given any $S(\Gamma)$, there is a space isotopy under which $S(\Gamma)$ is equivalent to a disk/band surface for a regular projection $p: \Gamma \to P$. For $\Gamma' \simeq \Gamma$ we can similarly relate to $S(\Gamma')$ a disk/band surface for a regular projection $p': \Gamma' \to P'$ and rotate the space such that $P' = P$. Hence we consider $S(\Gamma)$ and $S(\Gamma')$ as disk/band surfaces for a regular projection p and show that they can be made identical except perhaps in the number of half twists on bands. Since any two regular projections of a graph as well as regular projections of equivalent graphs are related by a finite sequence of the five moves in Fig. 6 and plane isotopies ([6] or [9]), and each of these moves and plane isotopies

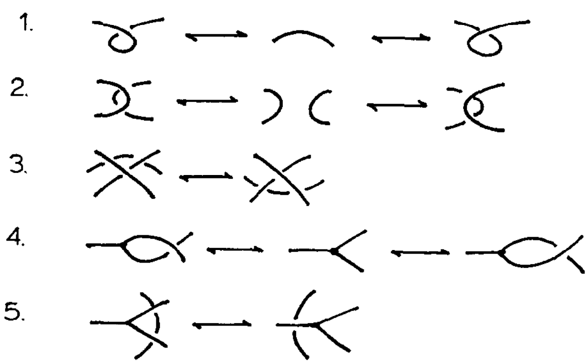


Fig. 6. Ambient isotopic and vertex moves.

extends (for 3-valent graphs) to take vertex disks to vertex disks, we have a space isotopy taking Γ' to Γ and $S(\Gamma')$ to a disk/band surface $S'(\Gamma')$ for the regular projection p which is identical to $S(\Gamma)$ except for the number of half twists on bands. But as we have seen the number of half twists on each band is uniquely determined by the requirement that the surface must have zero Seifert form, hence the two disk/band surfaces are ambient isotopic. This shows the existence and uniqueness of a good surface in the cases of θ-curves and K_4-graphs.

From the above discussion, our problem has been reduced to the existence and uniqueness of solutions to certain equation system. Each of the equations in the system corresponds to a relation $\langle x, y\rangle = 0$ for some x and y which are in a basis of $H_1(S(\Gamma))$.

Now we prove statement (2) of Theorem 2.4. Suppose Γ is an embedding of a connected trivalent abstract graph G with more than six edges and a prime planar embedding. Then the number of vertices in the graph is $2k$ and the number of edges is $3k$ for some integer $k \geqslant 3$. If $S(\Gamma)$ is a disk/band surface of Γ, then $\beta_1 = \dim H_1(S(\Gamma)) = k+1$. Then there are $(1/2)(k+1)(k+2)$ Seifert pairings of the homology generators of $S(\Gamma)$. Let n_i be the number of half twists on the band about edge a_i. Setting all the Seifert pairings equal to zero we get an equation system, with variables $n_1, \ldots, n_{3k}$.

We start with a planar embedding Γ_0, with oriented edges $a_1, \ldots, a_{3k}$. By Lemma 2.6, the orders of variables $n_1, \ldots, n_{3k}$ associated with edges $a_1, \ldots, a_{3k}$ can be arranged such that $a_1, \ldots, a_l$ are common edges of two boundary cycles; $a_{l+1}, \ldots, a_{3k}$ are boundary edges and the relations can be ordered by listing, first, all the self pairings of homology generators corresponding to boundary cycles without boundary edges, secondly, the Seifert pairings of generators corresponding to adjacent boundary cycles, thirdly, the self pairings of homology generators corresponding to boundary cycles with boundary edges, and finally, the pairings of disjoint boundary cycles. By ordering the edges and the pairings carefully, the matrix of the equation system $A(\Gamma)$ will be of the following form, with $m = (1/2)(k+1)(k+2)$ rows and $n = 3k$ columns. Since $k \geqslant 3$, $A(\Gamma)$ is not a square matrix. Write $m = h + 3k + g$. In the following presentation of $A(\Gamma)$, the submatrices H and G have at least three and two nonzero entries in each row, respectively, and 0 represents zero matrix.

$$A(\Gamma) = \begin{bmatrix} H_{h\times l} & 0_{h\times(3k-l)} \\ I_{l\times l} & 0_{l\times(3k-l)} \\ G_{(3k-l)\times l} & I_{(3k-l)\times(3k-l)} \\ 0_{g\times l} & 0_{g\times(3k-l)} \end{bmatrix}.$$

We prove part (i) by constructing an embedding $\Gamma(G)$ without a good surface. The discussion will be divided into two cases.

Case 1. If there exists a zero row in $A(\Gamma)$, then there must be a pair of disjoint cycles, say, x and y, in a planar embedding Γ_0 of the graph. Pick an edge a_i from x, and an edge a_j from y, such that a_i and a_j don't belong to the same boundary cycle. Lift them out of the plane, and make a full twist of the two edges as shown in Fig. 7. Let Γ be the resulting graph. We claim that there are no good disk/band surface $S(\Gamma)$ for Γ.

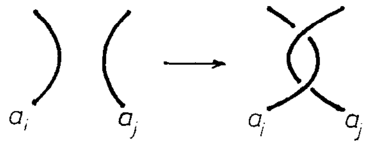


Fig. 7. A full twist of edges a_i and a_j.

Note that w_{ij} appears in a relation $\langle x, y\rangle = 0$ if and only if a_i and a_j are in the boundary cycles x and y, respectively. If the equations for the above embedding Γ is written into the form

$$A(\Gamma)_{m\times n}N_{n\times 1} = B(\Gamma)_{m\times 1},$$

then $B(\Gamma)$ will have all zero entries except for the $h+3k+j$ (some $j>0$) position, which makes rank $[A(\Gamma)\,|\,B(\Gamma)] >$ rank $A(\Gamma)$ so there is no solution to the equations and no good disk/band surface for Γ. Therefore, there are no good surfaces for Γ.

Case 2. If there are no zero rows in $A(\Gamma)$, then $h \neq 0$, so there must be at least one boundary cycle x in Γ_0 without boundary edge. Take a boundary cycle y in Γ_0 with a boundary edge a_j. Then they must have a unique common edge a_s because $g=0$. Let a_i be an edge in x which is different from a_s. By Lemma 2.6, a_i belongs to two boundary cycles, x and z. Let the common edge of y and z be a_t. We construct an embedding Γ by lifting a_i and a_j out of the plane and make a full twist as above. Then the $h+s$ and $h+t$ entries of $B(\Gamma)$ are nonzero (equal to ± 2, the sign depend on orientations of the four edges) and all the other entries of $B(\Gamma)$ are zero. Therefore, the solution must satisfy $n_1 = \cdots = n_{s-1} = n_{s+1} = \cdots = n_{t-1} = n_{t+1} = \cdots = n_l = 0$ and $n_s = \pm 2$, $n_t = \pm 2$. But since n_s appears in the self pairing of the boundary cycle x and n_t doesn't, there will be an integer p, $1 \leqslant p \leqslant h$, the (p, s) entry of $A(\Gamma)$ is one, and the (p, t) entry is zero. Also, the (p, q) entries will be zero for $q > 1$. This is contradictory because the pth equation in the system shows $n_s = 0$. Therefore, there is no solution to $A(\Gamma)N = B(\Gamma)$ for this embedding Γ of G.

Finally, we prove part (ii). Through the above discussion, we have seen that $A(\Gamma)$ is of full rank. If Γ has a disk/band surface with zero form for a certain regular projection of Γ, then the number of twists on bands is a solution to $A(\Gamma)N = B(\Gamma)$. In that case, the equation system has a solution and we conclude that rank $[A(\Gamma)\,|\,B(\Gamma)] =$ rank $[A(\Gamma)]$, and hence the solution is unique. Again, any two surfaces $S(\Gamma)$, $S'(\Gamma)$ are equivalent to each other except perhaps for twists; and the above argument shows that the numbers of twists are unique. □

Theorem 2.7. *Let G be a connected trivalent planar abstract graph. If the planar embedding $\Gamma_0(G)$ is not prime, then*

(1) *there exists an embedding Γ without good surface;*

(2) *there exists an embedding Γ with more than one good surface.*

Proof. We prove part (1) by first claiming that there must be a pair of disjoint boundary circles in Γ_0. Then, let a_i and a_j be edges in each of the cycles; we can make an embedding Γ by twisting them, as in the proof of Theorem 2.4. The linking number of these cycles are not zero (equal to $(1/2)w_{ij}$). It is clear that any twisting on bands along the edges cannot change the Seifert pairing of the two corresponding homology generators. Therefore, there cannot be any good surface about Γ.

Proof of the claim.

Case 1. If there exists a type I admissible circle C in $\mathbb{R}^2$ intersecting Γ_0 in a single point, then we can always assume that the circle C intersects Γ_0 in an interior point of an edge. Note that this edge does not belong to any boundary cycle. There must be a loop entirely inside the disk bounded by C, and a loop entirely outside the disk. Pick a boundary cycle in each of the regions bounded by the loops. They are disjoint boundary cycles.

Case 2. If there is no type I admissible circle, then there must be an admissible circle of type II intersecting Γ_0 in two points which are interior to two edges, respectively (Fig. 8).

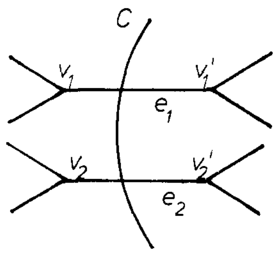


Fig. 8. The intersection of circle C with Γ_0.

It is known by definition that there is a path in the graph connecting v_1 and v_2, lying entirely in one side of C and there is another edge containing v_1 which is not in the path. If the other vertex of this edge is a vertex in the path, then a loop is obtained in one side of C. Otherwise, take one of the other two edges containing that vertex and look at the other vertex of that edge. If it is in the loop or is a previous vertex, a loop is obtained in one side of C. Continuing the process until a vertex or going back to a vertex in the "old path", a loop will eventually be obtained in one side of C. Pick a boundary cycle in the bounded region of the loop. The same discussion starting with v_1' and v_2' leads to a boundary cycle in the other side of C. They are disjoint boundary cycles.

We prove part (2) by considering the above two cases separately.

Case 1. There exists an admissible 2-sphere of type I. First, look at the special case where G is a "handcuff graph" (to illustrate the method), and let $\Gamma_0(G)$ and $\Gamma(G)$ be embeddings as shown in Fig. 9. Let $S_n(\Gamma)$ $(n \in \mathbb{Z})$ be as in Fig. 10.

Claim. *The* S_n *are all different* (*and all good*).

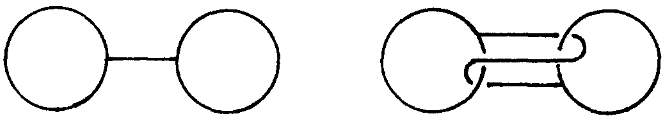


Fig. 9. Two embeddings, Γ_0 and Γ, of the handcuff graph.

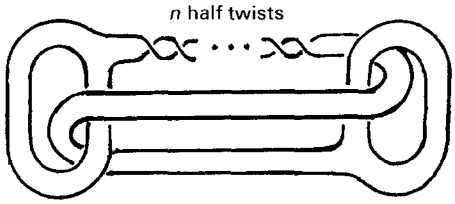


Fig. 10. A good surface $S_n(\Gamma)$ of Γ.

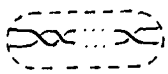
If two such surfaces are equivalent, then so are their boundary links. Let L_n denote the boundary link of S_n. We claim all these links are mutually inequivalent. Let $F_n = \langle L_n \rangle$, the Kauffman bracket polynomial of L_n (for definition, see [4, 5]) and $\hat{F} = \langle \circ\circ\circ\circ \rangle$. Through calculation we get

$$F_0 = A^{20} - A^{16} + A^{12} - A^{-4} + A^{-8} - A^{-12} + A^{-20} - A^{-24} + A^{-28},$$

$$F = \delta^3 = -A^6 - 3A^{-2} - 3A^{-2} - A^{-6},$$

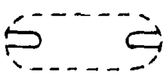
where $\delta = -A^2 - A^{-2}$. To finish the calculation we need the following lemma.

Lemma 2.8. *Let K_n be a link with a regular projection that includes a pair of strands, with n half twists, as shown below:*



K_n

Let $\hat{K}$ be the link obtained from K_n by replacing the tangle above with



$\hat{K}$

Then the normalized brackets are related as follows:

$$\langle K_n \rangle = A^m \langle K_{n-m} \rangle + \left(\sum_{k=1}^{|m|} (-1)^{n-k} A^{\mathrm{sign}(n)(-3|n|+4k-2)} \right) \langle \hat{K} \rangle$$

for $0 < m \leq n$ or $n \leq m < 0$.

Proof. Since $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$ and $\langle \times \rangle = A^{-1}\langle \asymp \rangle + A\langle)(\rangle$, the formula can be obtained by induction on n. $\square$

If we apply the lemma to our boundary links L_n with $m = n$ we know that the highest degree term in F_n is A^{n+20}, and for n sufficiently large the lowest degree term in F_n is $(-1)^n A^{-3n-4}$. Since $w(L_n) = -n$, the Kauffman polynomial (see [4, 5]) of L_n is

$$f_{L_n} = \alpha^{-w(L_n)}\langle L_n\rangle = (-A^3)^n \langle L_n\rangle,$$

with leading term $(-1)^n A^{4n+20}$. Hence, $L_n \neq L_m$ and $S_n \neq S_m$ for $n \neq m$.

Having completed the case of a handcuff graph, we now continue Case 1 of (2) with a general graph G. We shall show that $\Gamma_0(G)$ contains a particularly well-situated handcuff graph and use that handcuff to construct the desired embeddings of G. Let e_0 be the edge of $\Gamma_0(G)$ which intersects an admissible type I circle C, v_0 and v_0' be two vertices of e_0. We shall find a handcuff graph in $\Gamma_0(G)$ consisting of two boundary cycles joined by an arc through e_0. We first look at v_0. The other two edges containing v_0 either belong to the same boundary cycle (and might be identical) or both belong to no single boundary cycle. If they are in a boundary cycle, then we are done with half the desired handcuff. Otherwise, take one of the other two edges and look at the other vertex of that edge. Repeat this process until a "left-most" vertex v_1 (Fig. 11) can be found with the other two edges belonging to a boundary cycle (the two edges might be identical). Denote the edge containing v_1 which belongs to the cycle and "near" C by e_1 (Fig. 11). Then we start at v_0' and find a "right-most" vertex v_2 such that one of the three edges containing v_2, denoted by e_3, belongs to no boundary cycle, and the other two (possibly identical) belong to a boundary cycle (Fig. 11). Denote one of them by e_2 (see Fig. 11). (Note: it may happen that e_3 is e_0.)

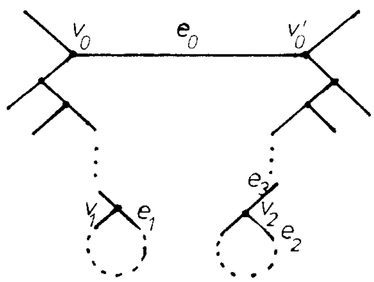


Fig. 11. Edges e_1, e_2 and e_3.

Take an embedding Γ as shown in Fig. 12, and the disk/band surface $S_n(\Gamma)$ with zero Seifert form as shown in Fig. 13. The link $L_n = \partial S_n(\Gamma)$ is shown in Fig. 14. As we showed in the special case of a handcuff, $L_n \neq L_m$ if $n \neq m$, hence $S_n \neq S_m$.

Case 2. If there is no type I circle, then there must be an admissible type II circle. Again, let us look at a special case where $\Gamma_0(G)$ is a "double θ-curve" with an ordering and an orientation to edges as in Fig. 15(a).

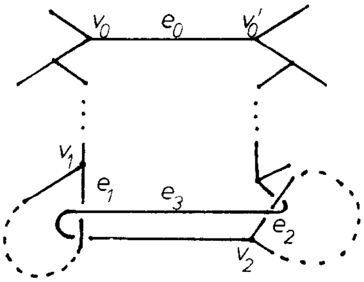


Fig. 12. The embedding Γ.

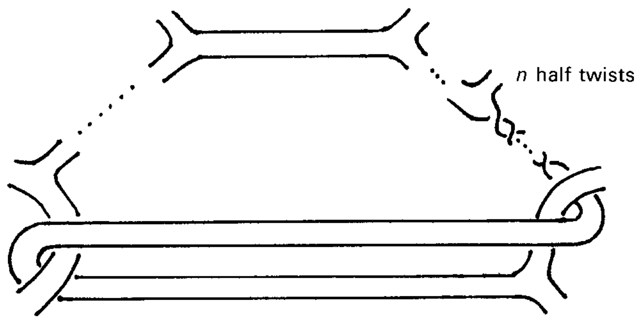


Fig. 13. A good surface $S_n(\Gamma)$ of Γ.

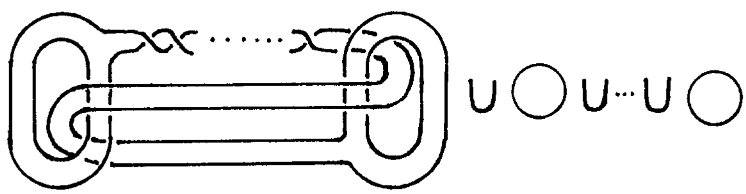


Fig. 14. The boundary link L_n of $S_n(\Gamma)$.

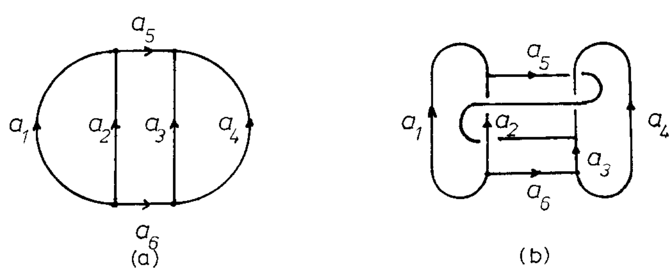


Fig. 15. Two embeddings of the double θ-curve with orientations and labelings of edges.

Consider the embedding Γ shown in Fig. 15(b). The numbers of twists n_i $(i = 1, \ldots, 6)$ on the bands of a disk/band surface of Γ with zero Seifert form should satisfy the following relations

$$n_2 = -w_{25} = 2,$$

$$n_3 = -w_{54} = -2,$$

$$n_1 = -n_2 = -2,$$

$$n_4 = -n_3 = 2,$$

$$n_5 + n_6 = -2w_{25} = 4.$$

Let $a = n_5$ and $b = n_6 = 4 - a$ and $S(\Gamma)_{a,b}$ be a disk/band surface about the above regular projection of Γ with

$$n_1 = -2,$$

$$n_2 = 2,$$

$$n_3 = -2,$$

$$n_4 = 2,$$

$$n_5 = a,$$

$$n_6 = 4 - a,$$

as shown in Fig. 16.

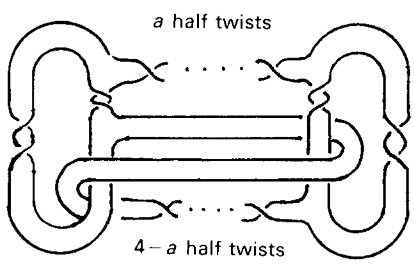


Fig. 16. A good surface $S(\Gamma)_{a,b}$ of Γ.

Let $L_{a,b}$ = boundary link of $S(\Gamma)_{a,b}$, $F_{a,b} = \langle L_{a,b} \rangle$. Take $(a, b) = (4, 0)$ and $(a', b') = (0, 4)$, apply Lemma 2.8, we get

$$F_{4,0} = A_4 F_{0,0} + \left(\sum_{k=1}^{3} (-1)^{3-k} A^{4k-10} \right) F_0,$$

$$F_{4,0} = A_4 F_{0,0} + \left(\sum_{k=1}^{3} (-1)^{3-k} A^{4k-10} \right) \delta^2,$$

since $F_0 \neq \delta^2$, $F_{4,0} \neq F_{0,4}$ and $S(\Gamma)_{4,0} \neq S(\Gamma)_{0,4}$.

In general, let e_1 and e_2 be the two edges that intersect the circle C in two points, v_i and v_i' be vertices of e_i, $i=1,2$. By hypothesis, e_1 and e_2 belong to exactly one boundary cycle. Let A be the open disk bounded by the cycle in $\mathbb{R}^2-\Gamma_0$ (Fig. 17). The other two edges containing v_1 other than e_1 must belong to some boundary cycle, so do the two edges containing v_1' other than e_1. Denote the open disks by B and D, respectively (Fig. 17), and let embedding Γ of G be as in Fig. 18. A set of disk/band surfaces $S(\Gamma)_a$ with zero Seifert form of this regular projection of Γ can be chosen as in Fig. 19. The boundary link $L(\Gamma)_a$ of the surface $S(\Gamma)_a$ is shown in Fig. 20. The results of the special case indicate $L(\Gamma)_4 \neq L(\Gamma)_0$, hence, $S(\Gamma)_4 \neq S(\Gamma)_0$. □

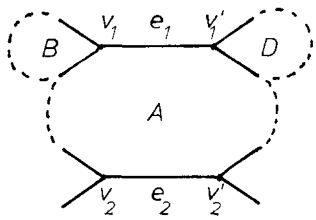


Fig. 17. Edges e_1 and e_2 intersecting circle C and disks A, B and D.

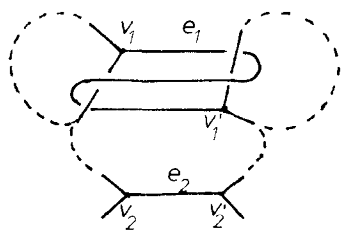


Fig. 18. The embedding Γ.

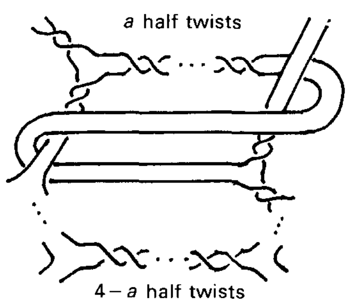


Fig. 19. A good surface $S(\Gamma)_a$ of Γ.

In conclusion, θ-curves and K_4-graphs are the only connected trivalent graphs that can be associated uniquely with an orientable surface with zero Seifert form, hence, a link of self linking number zero. Moreover, the link (in fact the surface) is an ambient isotopy invariant of the graph.

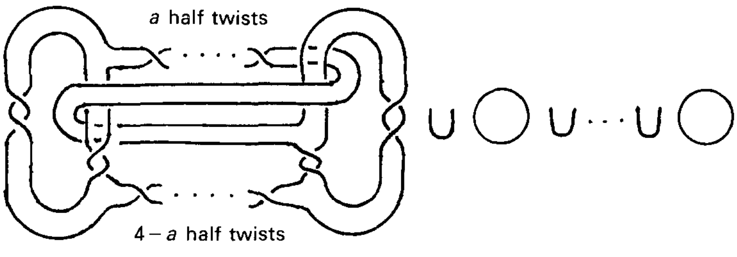


Fig. 20. The boundary link $L(\Gamma)_a$ of $S(\Gamma)_a$.

3. The invariant link as a knot-type detector

We have shown that a surface of zero Seifert form, and hence a link of zero self linking number, can be associated to an embedded θ-curve or K_4-graph as a knot type invariant of graphs. Furthermore, the link serves as part of a "peripheral coordinate system" of the graph complement in 3-space, in fact, the components of the link can be the "standard longitudes" of a tubular neighborhood of the embedded graph, in the sense that each one vanishes in $H_1(\mathbb{S}^3 - K)$. In particular, in the case of a nontrivial θ-curve, the link cannot be a Brunnian link: if every sublink of the link is trivial, then the link itself must be a trivial link. The reason is that one of the components of the link can be viewed as the result of a band operation of the other two components [10]. By a theorem of Scharlemann [10] we know that the band must be a trivial one, hence, the link is trivial. Furthermore, Scharlemann's theorem implies that if the associated link is trivial then the θ-curve is trivial (planar).

As a first application of this knot type invariant of graphs, it can be used to distinguish (unoriented) θ-curves through five crossings and K_4-graphs through four crossings as listed in [11]. To distinguish various of these graphs we end up needing to distinguish the associated links, and we do this using HOMFLY polynomials computed with a program for drawing links and calculating polynomials developed by R. Litherland.

Example 1. $\theta_{3,1}$ vs. $\theta_{5,2}$ (Fig. 21) (here $\theta_{i,j}$ are names for θ-graphs used in [11]). The constituent knots of $\theta_{3,1}$ and $\theta_{5,2}$ are both $\{3_1, 0, 0\}$ (where 0 means trivial knot). But the associated links are different, for the two trivial components in the first link form the link 5_1^2 and those in the second link form an 11-crossing link (Fig. 22), which is different from the link 5_1^2.

Fig. 21. $\theta_{3,1}$ and $\theta_{5,2}$.

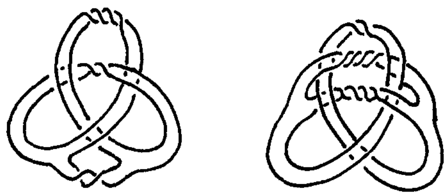


Fig. 22. The associated links of $\theta_{3,1}$ and $\theta_{5,2}$.

Example 2. $\theta_{5,6}$ vs. $\theta_{5,7}$ (Fig. 23). Both of them have constituent knots $\{5_2, 0, 0\}$, but the sublinks of the associated links consisting of two trivial components are different, as link 7_3^2 and a 10-crossing link (Fig. 24).

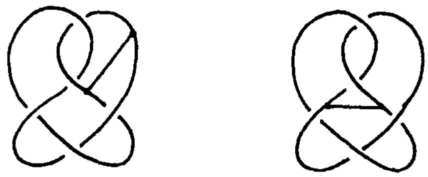


Fig. 23. $\theta_{5,6}$ and $\theta_{5,7}$.

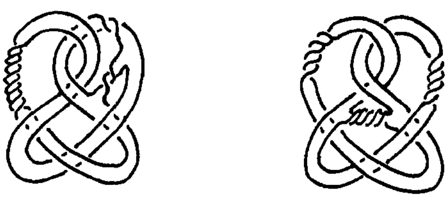


Fig. 24. The associated links of $\theta_{5,6}$ and $\theta_{5,7}$.

Example 3. Kinoshita's θ-curve (Fig. 1) is nontrivial (i.e., not planar), even though all its constituent knots are trivial. The associated link, as pictured in Fig. 25, is nontrivial since any one of its 2-component sublinks is a 7_3^2 link.

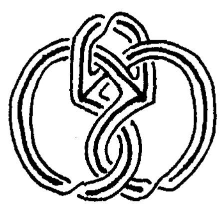


Fig. 25. Kinoshita's θ-curve and its associated link.

In order to understand how the invariant works as a knot type detector, we want to see whether the process of "graph → special surface → link" can be "reversed", and first of all, we want to know whether the special surface determines its graph spine.

3.1. Theta-curve spines are uniquely determined by the surfaces

Lemma 3.1. *Let K be a θ-curve spine of a standard (planar) disk with two holes P_0, then there is an isotopy of P_0, fixing boundary of $P_0(\partial P_0)$ and sending K to the standard θ-curve spine K_0 of P_0 (Fig. 26).*

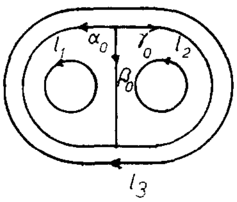


Fig. 26. K_0 and P_0.

Proof. Let l_i $(i = 1, 2, 3)$ be the (oriented) boundary components of P_0 as shown in Fig. 26. If K is a θ-curve spine of P_0, then any constituent knot in K is a simple closed curve in P_0 which is homotopically nontrivial, so it is freely homotopic to some $l_i^{\pm 1}$, and the three constituent knots of K are not homotopic in P_0. Therefore K can be ordered and oriented such that (refer to Fig. 26) $\alpha - \beta \approx l_1$, $\beta - \gamma \approx l_2$ and $\gamma - \alpha \approx l_3$ (where "$\approx$" means "freely homotopic to"). Since homotopic simple closed curves on a surface are isotopic [2], there is an isotopy rel ∂P_0 taking $\gamma - \alpha$ to $\gamma_0 - \alpha_0$. With an additional isotopy we may assume $\alpha \to \alpha_0$ and $\gamma \to \gamma_0$. Let β' be the image of β under the isotopy. Let A be the annulus bounded by $\gamma_0 - \alpha_0$ and ∂P_0, and P_1 be the twice punctured disk $(\overline{P_0 - A})$. Since $\alpha_0 - \beta'$ is homotopic to l_1, there is a homeomorphism h of P_1 fixing ∂P_1 and taking β' to β_0. This homeomorphism is equivalent to h_{2n} for some $n \in \mathbb{Z}$ via an isotopy fixing ∂P_1 [1], where h_{2n} is the $2n$-multiple of the homeomorphism h_1 of P_1 which sends l_1 to l_2, l_2 to l_1, and is identity outside a disk in the interior of P_1 including l_1 and l_2. So β is isotopic rel ∂P_1 to $B_n = h_{2n}(\beta_0)$ (Fig. 27).

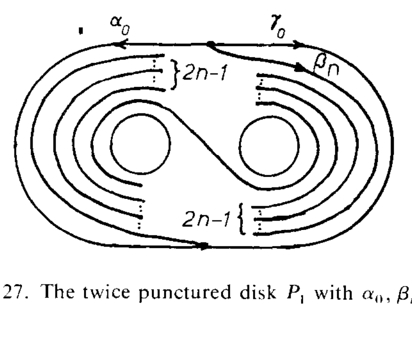


Fig. 27. The twice punctured disk P_1 with α_0, β_n, γ_0.

Let N be a small cylindrical neighborhood of $\alpha_0 \cup \gamma_0$ in the interior of P_0, which is parametrized by (y, θ), with $-1 \leq y \leq 1$, $0 \leq \theta \leq 2\pi$, where $\alpha_0 \cup \gamma_0$ is described by $y = 0$, and the two vertices of K by $(1, 0)$ and $(-1, 0)$, as in Fig. 28. We define a map $g_1 : P_0 \to P_0$ by the rule that if a point is in N, then its image is given by:

$$g_1 : (y, \theta) \to \begin{cases} (y, \theta + 2\pi y), & \text{if } 0 \leq y \leq 1, \\ (y, \theta - 2\pi y), & \text{if } -1 \leq y \leq 0, \end{cases}$$

while all points of $P_0 - N$ are fixed (Fig. 28). Let g_n be the nth power of g_1. Then it is an isotopy of $P_0 \operatorname{rel} \partial P_0$. This isotopy followed by an isotopy of P_1 fixing ∂P_1 takes K to K_0. □

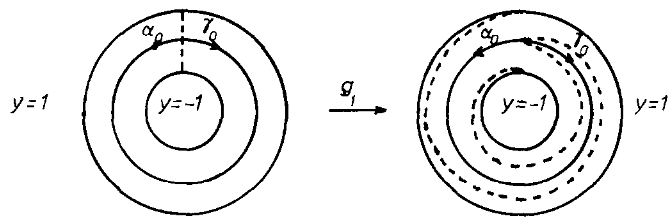


Fig. 28. The map g_1 restricted on N.

Corollary 3.2. *Let P_0 be the standard twice punctured disk with boundary components l_1, l_2, l_3, each of them inherited an orientation from P_0. If K is an ordered and oriented θ-curve spine of P_0 such that $\alpha - \beta \approx l_1$, $\beta - \gamma \approx l_2$ and $\gamma - \alpha \approx l_3$, then there is an isotopy of P_0 rel ∂P_0 taking K to K_0.*

Proposition 3.3. *Any two θ-curve spines, K_1 and K_2, of an embedded twice punctured disk P are ambient isotopic.*

Proof. Let h be the embedding $P_0 \to \mathbb{R}^3$ (or $\mathbb{S}^3$), $P = h(P_0)$, and H_t be the isotopy of $P_0 \operatorname{rel} \partial P_0$ taking one θ-curve spine $h^{-1}(K_1)$ to another θ-curve spine $h^{-1}(K_2)$ of P_0. Then the composition $h \circ H_t \circ h^{-1}$ is an isotopy of P fixing ∂P and taking K_1 to K_2. Since P is compact, $h \circ H_t \circ h^{-1}$ can be extended to an isotopy of $\mathbb{R}^3$ (or $\mathbb{S}^3$) taking K_1 to K_2 [3]. □

Corollary 3.4. *Let P be an embedded twice punctured disk, $\partial P = \{l_1, l_2, l_3\}$, each of l_i has the boundary orientation inherited from an orientation of P. Then any two ordered and oriented θ-curve spines K_1 and K_2 of P with $\alpha_i - \beta_i \approx l_1$, $\beta_i - \gamma_i \approx l_2$ and $\gamma_i - \alpha_i \approx l_3$ $(i = 1, 2)$ are ambient isotopic via an isotopy respecting the ordering and orientation.*

3.2. An example of two different K_4-graphs being spines of a surface with zero Seifert form

Let S be the surface in Fig. 29 and Γ_1 be a "standard" spine of S (see Fig. 29). The set of seven constituent knots of Γ_1 consists of four figure-8 knots, two connected sums of figure-8 knots, and a trivial knot.

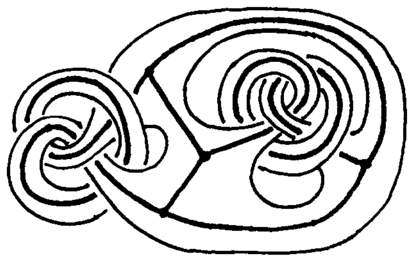


Fig. 29. Surface S and its spine Γ_1.

Let Γ_2 be another spine of S (Fig. 30). The set of constituent knots of Γ_2 consists of four figure-8 knots, one connected sum of two figure-8 knots, and knots k_1 and k_2 as in Fig. 31.

We claim that both k_1 and k_2 are not trivial, and then conclude that $\Gamma_1 \neq \Gamma_2$.

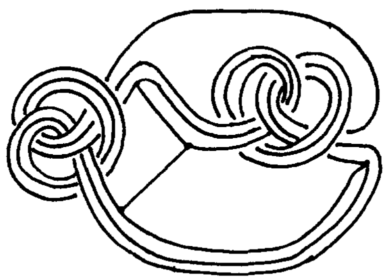


Fig. 30. Another spine Γ_2 of S.

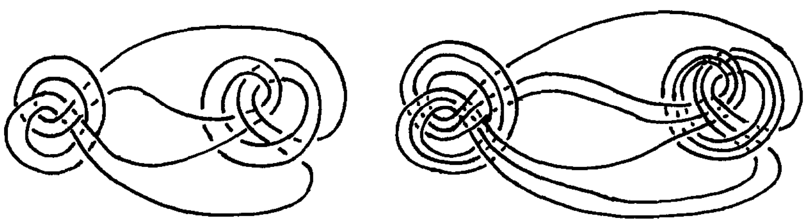


Fig. 31. Knots k_1 and k_2.

Claim 1. *The figure-8 knot is a proper companion of* k_1.

Consider the following knot, k_1', in a standard solid torus V (Fig. 32). In the universal covering of V, k_1' is lifted to the link in Fig. 33. Any two adjacent components of the above link are geometrically linked since the link in Fig. 34 is not splittable (being two parallel copies of a nontrivial knot). Therefore, k_1' is geometrically essential in V, and hence the figure-8 knot is a proper companion of k_1.

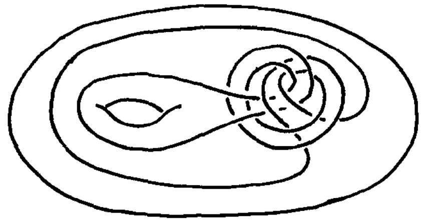


Fig. 32. k_1' in the solid torus V.

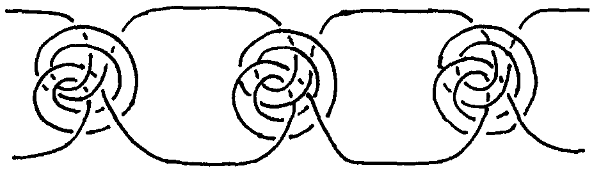


Fig. 33. Lifting of k_1'.

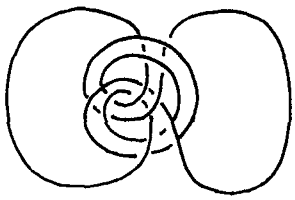


Fig. 34. A nonsplittable link.

Claim 2. *The figure-8 knot is a proper companion of* k_2.

Consider the following knot, k_2', in a standard torus V (Fig. 35). Since the algebraic intersection number of k_2' and the meridian disk of V are 1 for appropriate orientation of k_2', k_2' is geometrically essential in V, and hence the figure-8 knot is a companion of k_2. The referee pointed out another proof by computing HOMFLY polynomials.

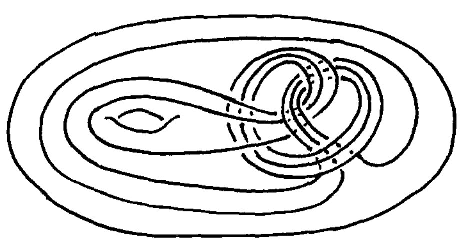


Fig. 35. k_2' in solid torus V.

The "first term" of each polynomial was computed by the referee as the following:

$$P_{k_1}(v, z) = (-v^{-8} + 4v^{-6} - 8v^{-4} + 12v^{-2} - 13 + 12v^2 - 8v^4 + 4v^6 - v^8) + \cdots$$

and
$$P_{k_2}(v, z) = (4v^{-14} - 27v^{-12} + 83v^{-10} - 141v^{-8} + 108v^{-6} + 82v^{-4} - 341v^{-2}$$
$$+ 82v^4 + 108v^6 - 141v^8 + 83v^{10} - 27v^{12} + 4v^{14}) + \cdots$$

4. Discussion on graphs with vertex of higher valence

If the graph G contains at least one vertex of valence ≥ 4, then there are different good surfaces for some ambient isotopic embeddings. Moreover, we can find an infinite sequence of ambient isotopic embeddings with different good surfaces. This result is stated in the following theorem.

Theorem 4.1. *Let G be a connected planar abstract graph with at least one vertex of valence ≥ 4, then there are equivalent embeddings of G with mutually different good surfaces.*

Proof. The discussion will be separated into two cases, depending on whether or not the graph admits a certain kind of splitting.

Case 1. If there is an admissible circle of type I intersecting Γ_0, a planar embedding of G, in an interior point of an edge, then the discussion is similar to that in the proof of part (2) of Theorem 2.7.

Case 2. Otherwise.

First consider a special example of Case 2, where G is the figure-8 graph (Fig. 36(a)). Let the embeddings Γ_0 and Γ_n be as in Fig. 36(a) and (b). Note these are ambient isotopic embeddings. A good surface $S(\Gamma_n)$ of Γ_n is defined as in Fig. 37. The boundary link L_n of $S(\Gamma_n)$ consists of two trivial knots and a "twist knot" k_n with n twists. If $n_1 \neq n_2$, then $k_{n_1} \neq K_{n_2}$, hence $L_{n_1} \neq L_{n_2}$, $S(\Gamma_{n_1}) \neq S(\Gamma_{n_2})$, although Γ_{n_1} and Γ_{n_2} are ambient isotopic embeddings.

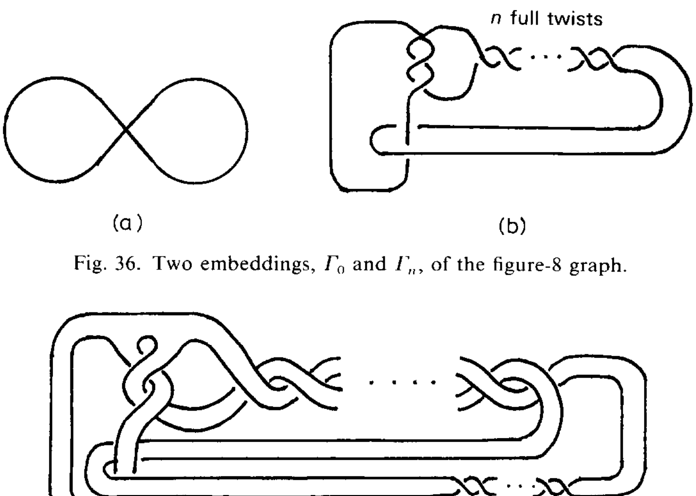


Fig. 36. Two embeddings, Γ_0 and Γ_n, of the figure-8 graph.

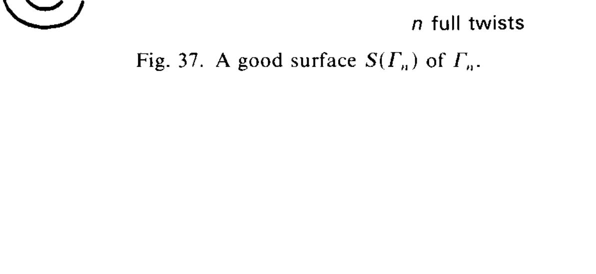


Fig. 37. A good surface $S(\Gamma_n)$ of Γ_n.

Now let G be an arbitrary graph of Case 2, and let v_0 be a vertex of G of valence ≥ 4. By hypothesis, every edge in Γ_0 must belong to some boundary cycle. Without loss of generality we may assume that there are two adjacent edges of Γ_0 containing v_0 which don't belong to the same boundary cycle (Fig. 38(a)). Let Γ_n be as in Fig. 38(b). The surface $S(\Gamma_n)$ can be constructed in the same way as for the knotted figure-8 graph. $L_n = \partial(S(\Gamma_n))$ consists of trivial components and a "twist knot" k_n. Hence, if $n_1 \neq n_2$, then $S(\Gamma_{n_1}) \neq S(\Gamma_{n_2})$. But Γ_{n_1} is equivalent to Γ_{n_2}. □

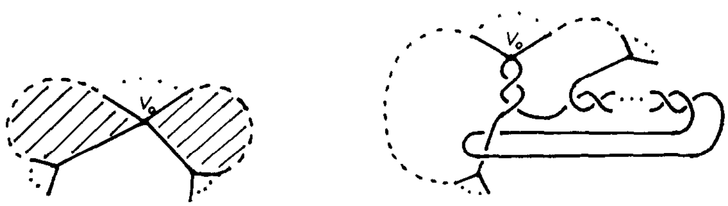


Fig. 38. Two embeddings, Γ_0 and Γ_n, of the graph G.

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