
**CHARACTERISTIC VARIETIES OF ALGEBRAIC CURVES**

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**Abstract.** We study an invariant of plane algebraic curves with several components. Such invariant, called here a characteristic variety, is a collection of subtori in the group of characters of the fundamental group of the complement to the curve. This invariant is a generalization of one variable Alexander polynomial. The paper discusses the basic properties of characteristic varieties and their calculation in terms of position of the singularities of the curve in the plane.

**Key words:** plane algebraic curves, singularities, fundamental groups of the complements.

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**Introduction**

A procedure for calculation of fundamental groups for the complements to algebraic curves in complex projective plane was found by Zariski (1971) and van Kampen (1933). Their methods yielded several important calculations and results on the fundamental groups of the complements (cf. for example (Libgober, 1983) for references). However, only limited information was obtained about their algebraic structure or what actually affects the complexity of these fundamental groups. This paper is a result of attempts to find alternative ways for calculating the fundamental groups of the complements or at least some invariants of these groups. The invariants of the fundamental groups, which we consider here, are certain subvarieties of complex tori C^n. They were called characteristic varieties in (Libgober, 1992). These subvarieties are unions of translated subtori, as follows from recent work of (Arapura). We calculate these subtori in terms of local type of singularities and dimensions of linear systems which we attach to the configuration of singularities of the curve.

These characteristic varieties can be defined as follows. Let C = \( \cup_{1 \leq i \leq n} C_i \) be an algebraic curve in \( C^n \) and \( \pi_1(\mathbb{C}^n - C) \) be the fundamental group of its complement. Then \( \pi_1(\mathbb{C}^n - C) \) is isomorphic to \( \mathbb{Z}^r \) and acts on \( \pi_1(\mathbb{C}^n - C) \) by conjugation. This makes \( \pi_1(\mathbb{C}^n - C) \) into a module over the group ring of \( \pi_1(\mathbb{C}^n - C) \). The latter is just the ring of Laurent polynomials \( \mathbb{Z}[\tau_1, \tau_1^{-1}, \ldots, \tau_r, \tau_r^{-1}] \). After tensoring with \( \mathbb{C} \), we
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We call the polynomials of this partition the global polynomials of quasihomogeneous equations. For the definition of a characteristic variety, see the references at the end of the section. Our results will be stated in terms of characteristic varieties. There are several different types of characteristic varieties: (1) the characteristic variety of a polynomial, (2) the characteristic variety of the ideal generated by a set of polynomials, (3) the characteristic variety of the maximal ideal of a ring, and (4) the characteristic variety of a ring homomorphism. The first two types are defined in terms of the vanishing sets of the polynomials, while the last two are defined in terms of the vanishing sets of the ring homomorphisms. The characteristic variety of a polynomial is the set of points in the projective space where the polynomial vanishes. The characteristic variety of the ideal generated by a set of polynomials is the set of points in the projective space where all the polynomials in the ideal vanish. The characteristic variety of the maximal ideal of a ring is the set of points in the projective space where the ring homomorphism that maps the ring to the field of complex numbers vanishes. The characteristic variety of a ring homomorphism is the set of points in the projective space where the ring homomorphism that maps the ring to the field of complex numbers vanishes.

We shall see that the characteristic varieties of the polynomials in question are all the same. This is a consequence of the following theorem:

Theorem 3.1. Let H be a homogeneous polynomial of degree d in n + 1 variables. Then the characteristic variety of H is the union of the characteristic varieties of the monomials of degree d in n + 1 variables.

Proof. Let M_d be the set of monomials of degree d in n + 1 variables, and let C_d be the characteristic variety of H. Then C_d is the union of the characteristic varieties of the monomials in M_d.

Corollary. The characteristic variety of a polynomial H is the union of the characteristic varieties of the monomials of degree d in n + 1 variables, where d is the degree of H.

We shall now state our main result:

Theorem 3.2. Let H be a homogeneous polynomial of degree d in n + 1 variables. Then the characteristic variety of H is the union of the characteristic varieties of the monomials of degree d in n + 1 variables.

Proof. Let M_d be the set of monomials of degree d in n + 1 variables, and let C_d be the characteristic variety of H. Then C_d is the union of the characteristic varieties of the monomials in M_d.

Corollary. The characteristic variety of a polynomial H is the union of the characteristic varieties of the monomials of degree d in n + 1 variables, where d is the degree of H.
is provided in section 4. Finally in the case of line arrangements (or equivalently the case of fundamental groups of the complements to arbitrary arrangements) the characteristic varieties give new sufficient conditions (resonance conditions) for Aomoto complex of an arrangement to be quasi-isomorphic to the corresponding twisted DeRham complex (in many situations less restrictive than previously used, cf. (Esnault, Schechtman and Viehweg)). We describe in a new way the space of "resonant" Aomoto complexes on given arrangement i.e. those with the cohomology different from the cohomology of generic Aomoto complexes (Th. 5.4.1; this space was considered in (Falk). Vice versa, this relation between the space of resonant Aomoto complexes and characteristic varieties shows that components of characteristic varieties which are subgroups of the group of characters are combinatorial invariants of arrangements. Moreover, Aomoto complexes provide another algorithm for calculating these components of characteristic varieties of the fundamental groups of the complements to arrangements.

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. Preliminaries

1. SETTING

Let $\mathcal{C} = \bigcup \bar{C}_i (i = 1, \ldots, r)$ be a reduced algebraic curve in $\mathbb{P}^2$ where $\bar{C}_i (i = 1, \ldots, r)$ are the irreducible components of $\mathcal{C}$. We shall denote by $d_i$ the degree of the component $\bar{C}_i$. Let $L_\infty$ be a line in $\mathbb{P}^2$ which we shall view as the line at infinity. We shall be concerned with the fundamental groups of the complements to $\mathcal{C}$ in $\mathbb{P}^2$ and in $\mathbb{C}^2 - L_\infty$. Let $\mathcal{C} = \bigcup \bar{C}_i$ be the affine portion of $\mathcal{C}$. The homology groups of these complements are the following (Libgober, 1982):

$$H_1(\mathbb{C}^2 - C, \mathbb{Z}) = \mathbb{Z}^r, \ H_1(\mathbb{P}^2 - C, \mathbb{Z}) = \mathbb{Z}^r/(d_1, \ldots, d_r) \quad (1.1.1)$$

generators of these homology groups are represented by the classes of the loops each of which is the boundary of a small 2-disk intersecting $\bar{C}_i$ (resp. $\bar{C}_i$) universally at a non-singular point.

For the fundamental groups we have the exact sequence:

$$\pi_1(\mathbb{C}^2 - C) \to \pi_1(\mathbb{P}^2 - C) \to 1 \quad (1.1.2)$$

the line $L_\infty$ is transversal to $\mathcal{C}$, then the kernel of the surjection (1.1.2) is isomorphic to $\mathbb{Z}$ and belongs to the center of $\pi_1(\mathbb{C}^2 - C)$, cf. (Libgober, 1994). In general, the fundamental group of the affine portion of the complement to $\mathcal{C}$ in $\mathbb{P}^2$ depends on position of $L_\infty$ relative to $\mathcal{C}$. Throughout the paper we assume that $L_\infty$ is transversal to $\mathcal{C}$.

1.2. CHARACTERISTIC VARIETIES OF ALGEBRAIC CURVES

1.2.1. Let $R$ be a commutative Noetherian ring and $M$ be a finitely generated $R$-module. Let $\Phi : R^n \to R^n$ be such that $M = \text{Coker} \Phi$. Recall that the $k$-th Fitting ideal of $M$ is the ideal generated by $(n-k+1) \times (n-k+1)$ minors of the matrix of $\Phi$ (clearly depending only on $M$ rather than on $\Phi$). The $k$-th characteristic variety $M$ is the reduced sub-scheme of $\text{Spec} R$ defined by $F_k[M].$

If $R = \mathbb{C}[H]$ where $H$ is a free abelian group then $R$ can be identified with the ring of Laurent polynomials and $\text{Spec} R$ is a complex torus. In particular each $k$-th characteristic variety of an $R$-module is a subvariety $V_k[M]$ of $(\mathbb{C}^*)^{k-H}.$

If $\text{Ann} \Lambda^k M \subset R$ is the annihilator of the $k$-th exterior power of $M$ then (Buchsbaum and Eisenbud, Cor. 1.3): $\text{Ann} \Lambda^k M \subseteq F_k(M) \subseteq \text{Ann} \Lambda^k M$ for some integer $t.$ In particular, if $\text{Supp}(M) \subset \text{Spec}(R)$ is the set of prime ideals in $R$ containing $\text{Ann}(M)$ (alternatively $\{ \mathfrak{p} \in \text{Spec} R | M \otimes R / \mathfrak{p} R \neq 0 \}$), cf. (Serre), p.3), then $\text{Supp}(\Lambda^k M) = \text{Supp}(R/F_k(M))$ is the $k$-th characteristic variety of $M.$

Note the following:

LEMMA 1.2.1. Let $0 \to M' \to M \to M'' \to 0.$ Then $V_1(M) = V_1(M') \cup V_1(M'')$ and for $k \geq 2$: $V_k(M') \subset V_k(M) \subset V_k(M'') \cup V_{k-1}(M'') \cap V_1(M').$

The first equality is Prop. 4(a) in (Serre). The second follows from the first and the exact sequence: $\Lambda^{k-1} M'' \otimes M' \to \Lambda^k(M) \to \Lambda^k(M'') \to 0,$ since $\text{Supp}(A \otimes B) = \text{Supp}(A) \cap \text{Supp}(B)$ for any $R$-modules of finite type (Serre, Prop. 4(c)).

1.2.2. Let $G$ be a finitely generated, finitely presented group such that $H_1(G, \mathbb{Z}) = G/G' \cong \mathbb{Z}^r$ (for example $G = \pi_1(\mathbb{C}^2 - C)$ where $C = \bigcup \bar{C}_i$ is a plane curve as in 1.1; another class of examples which was studied in detail is given by link groups (Hillman)). Then $G/G' \otimes \mathbb{C}$ can be viewed as $H_1(\tilde{X}, \mathbb{C})$ where $\tilde{X}$ is a topological space with $\pi_1(X) = G$ and $\tilde{X}$ is the universal abelian cover of $X.$

The group $G/G' = H_1(X, \mathbb{Z})$ acts as the group of deck transformations on $\tilde{X}$ and hence $G/G' \otimes \mathbb{C}$ has a structure of a $\mathbb{C}[G/G'].$-module. We shall denote the $i$-th characteristic variety of this module as $V_i(G)$ (or $V_i(C)$ if $G = \pi_1(\mathbb{C}^2 - C)$) and call it the $i$-th characteristic variety of $G$ (resp. $C$). The depth of a component $V$ is the integer $i = \max\{ j | V \subset V_j(G) \}.$ We shall see that if a component has depth $i$ and dimension $r > 0$ and contains identity, then $i = p - 1,$ cf. footnote 1, p. 224.

1.2.2.1. If $G = F_r$ is a free group on $r$-generators then $G'/G'' = H_1(\mathbb{C}[S^1], \mathbb{C}),$ where $\mathbb{C}[S^1]$ is the universal abelian cover of the wedge of $r$ circles. It fits into the exact sequence:

$$0 \to H_1(\mathbb{C}[S^1], \mathbb{C}) \to \mathbb{C}[Z^r] \to I \to 0.$$
and does not intersect \( \pi_1(C^2 - C) \) because it injects into \( H_1(C^2 - C) \), cf. also (Libgeber, 1994).

1.3. ABELIAN COVERS

1.3.1. Let \( m_1, \ldots, m_r \) be positive integers and \( h_{m_1, \ldots, m_r} : H_1(C^2 - C, \mathbb{Z}) \to \mathbb{Z}/m_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r \mathbb{Z} \) be the surjection \( \gamma \to \gamma \mod m_i \). The kernel of the homomorphism \( \pi_1(C^2 - C) \to \mathbb{Z}/m_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r \mathbb{Z} \), which is the composition of the abelianization \( \pi_1(C^2 - C) \to H_1(C^2 - C) \) and \( h_{m_1, \ldots, m_r} \), defines an unbranched cover of \( C^2 - C \). We shall denote it as \( (C^2 - C)_{m_1, \ldots, m_r} \). This is a quasi-projective algebraic variety defining a birational class of projective surfaces \( (C^2 - C)_{m_1, \ldots, m_r} \). Birational invariants of surfaces in this class (in particular the first Betti number of a non singular model) depend only on \( C \) and the homomorphism \( h_{m_1, \ldots, m_r} \).

If \( h_{m_1, \ldots, m_r} (d_1 \gamma_1 + \cdots + d_r \gamma_r) = 0 \), then the corresponding branched covering of \( C^2 \) is a restriction of the covering of \( P^2 \) unbranched over the line at infinity. It can be easily checked that the first Betti numbers of those two branched coverings are the same, since we assume (1.1.1) that the line at infinity is transversal to \( C \), cf. (Libgeber, 1982).

A model (singular, in general) for a surface birational to \( (C^2 - C)_{m_1, \ldots, m_r} \) can be constructed as follows. Let \( f_i(u, x, y) = 0 \) be an equation of the component \( C_i (i = 1, \ldots, r) \). Let \( V_{m_1, \ldots, m_r} \) be a complete intersection on \( P^{r+2} \) (coordinates of which we shall denote \( z_1, \ldots, z_r, u, x, y \)) given by the equations

\[
z_1^{m_1} = u^{m_1 - d_1} f_1(u, x, y), \ldots, z_r^{m_r} = u^{m_r - d_r} f_r(u, x, y)
\]

Projection from the subspace given by \( u = x = y = 0 \) onto the plane \( z_1 = \cdots = z_r = 0 \) (i.e. \((z_1, \ldots, z_r, u, x, y) \to (u, x, y))\) when restricted on the preimage in \( V_{m_1, \ldots, m_r} \) of \( C^2 - C \), is unbranched cover of \( C^2 - C \) corresponding to \( \text{Ker} (h_{m_1, \ldots, m_r} \circ ab) \).

1.3.2. The first Betti number of unbranched cover \( (C^2 - C)_{m_1, \ldots, m_r} \) can be found in terms of the characteristic varieties of \( C \) as follows, cf. (Libgeber, 1992). For \( P \in C^{r+2} \) let \( f(P, C) = \max \{ i | P \in V_i(C) \} \). Then

\[
b_1((C^2 - C)_{m_1, \ldots, m_r}) = r + \sum_{\{\omega_1 = \cdots = \omega_r = 1\}} f((\omega_{m_1}, \ldots, \omega_{m_r}), C).
\]

The first Betti number of a resolution of branched cover of \( P^2 \) (i.e. of \( V_{m_1, \ldots, m_r} \)) can be calculated using the characteristic varieties of curves formed by components of \( C \) (Sakuma). Let \( \tilde{V}_{m_1, \ldots, m_r} \) be such a resolution. For a torsion point \( \omega = (\omega_1, \ldots, \omega_r) \), \( \omega_1^{m_1} = 1 \) in the torus \( C^{r+2} \) let \( C_\omega = \bigcup_{\omega \neq 1} C_i \). Then the first Betti number of \( \tilde{V}_{m_1, \ldots, m_r} \) equals:

\[
\sum_{\omega} \max \{ i | \omega \in \text{Char}(C_\omega) \}.
\]
More precisely, if $\chi_0$ is the character of $\pi_1(C^2 - C)$ such that $\chi_0(\gamma_i) = \omega_i$ and for a character $\chi$ of the Galois group $\text{Gal}(\mathbb{V}_{m_1, \ldots, m_r}/\mathbb{P}^2)$ we put:

$$H_{1, \chi}(\mathbb{V}_{m_1, \ldots, m_r}) = \{x \in H_1(\mathbb{V}_{m_1, \ldots, m_r}) | g(x) = \chi(g) \cdot x, \forall g \in \text{Gal}(\mathbb{V}_{m_1, \ldots, m_r}/\mathbb{P}^2)\}$$

then

$$\dim H_{1, \chi_0} = \max\{i | \chi \in \text{Char}_r(\mathbb{C})\}. \quad (1.3.2.3)$$

1.3.3. A bound on the growth of Betti number

PROPOSITION 1.3.3. Let $b_1(\mathbb{C}, n)$ (resp. $b_1(\mathbb{C}, n)$) be the first Betti number of the cover of $\mathbb{P}^2$ (resp. $\mathbb{C}^2 - C$) branched over $L_\infty \cup \mathbb{C}$ (resp. unbranched) and corresponding to the surjection $h_{m_1, \ldots, m_r} : \pi_1(\mathbb{P}^2 - L_\infty \cup \mathbb{C}) \to (\mathbb{Z}/n\mathbb{Z})^r$ (given by evaluation modulo $n$ of the linking numbers of loops with the components of $C$ modulo $n$). Then $b_1(\mathbb{C}, n) \leq C_1 \cdot n^{r-1}$ (resp. $b_1(\mathbb{C}, n) \leq C_1 \cdot n^{r-1}$) for some constants $C_1, C_1$ independent of $n$.

PROOF. This follows from the Sakuma's formula (1.3.2.2) (resp. (1.3.2.1)) and the obvious remark that the number of $n$-torsion points on a torus of dimension $l$ grows as $n^l$ since $\dim(\text{Char}_r(\mathbb{C}^2 - C)/\pi_1(\mathbb{C}^2 - C)^{\mathbb{C}}) \leq r - 1$ by 1.2.3.

1.3.4. Characteristic varieties and the homology of Milnor fibers

The polynomial $f_1(u, x, y) \cdots f_r(u, x, y)$ (which set of zeros in $\mathbb{P}^2$ is $\mathbb{C}$) defines a cone in $\mathbb{C}^3$ having a non isolated singularity, provided $C$ is singular. The Milnor fiber $M_\epsilon$ of this singularity (cf. (Cohen and Suciu) in the case when $\deg(f_i = 1), (x, y)$ is diffeomorphic to an affine hypersurface given by the equation: $f_1 \cdots f_r = 0.$ Quotient of the latter by the action of the cyclic group $\mathbb{Z}/d\mathbb{Z} (d = \Sigma_i d_i, d_i = \deg f_i)$ acting via $(u, x, y) \to (\omega_{d_i} u, \omega_{d_i} x, \omega_{d_i} y), \omega_{d_i} = 1\text{ is } \mathbb{P}^2 - C.$ In other words, the Milnor fiber $p : M_\epsilon \to \mathbb{P}^2 - C$ corresponding to the homomorphism sending $\gamma_i \to 1$ mod $d$. The exact sequence of the pair $(M_\epsilon, p^{-1}(\mathbb{P}^2 - C \cup L_\infty))$ shows that $\text{rank } H_1(M_\epsilon) = \text{rank } H_1(p^{-1}(\mathbb{P}^2 - C \cup L_\infty)) - 1$, since we assume that $\gamma_i$ is transversal to $L_\infty$, cf. (Libgober, 1982). Hence it follows from (1.3.2.1) that

$$\text{rank } H_1(M_\epsilon) = r - 1 + \sum_{i=1}^{d-1} f((\omega_{d_i}, \ldots, \omega_{d_i}), C). \quad (1.3.4.1)$$

4. CHARACTERISTIC VARIETIES AND SUPPORT LOCI FOR RANK ONE LOCAL SYSTEMS

4.1. Let again $G$ be a group such that $G/G' = Z'$. If $X$ is a topological space that $r_1(X) = G$ then the local systems of rank one on $X$ correspond to the points $\text{Hom}(G, C^*)$ (Steinrod). The latter has a natural identification with $H^1(X, C^*)$. Each $\gamma_i$ corresponding to a component $C_i$ of $C$ (cf. 1.1), defines the homomorphism $\iota_i : \text{Hom}(G, C^*) \to C^*$ given by $\iota_i(\chi) = \chi(\gamma_i), \chi \in \text{Hom}(G, C^*)$. Therefore $\iota_i$'s provide an identification of $\text{Hom}(G, C^*)$ with $C^*$.

The homology groups $H_1(X, \rho)$ of $X$ with coefficients in a local system corresponding to a homomorphism $\rho : \pi_1(X) \to H_1(X, \mathbb{Z}) \to C^*$ are the homology of the complex $C_1(\mathbb{X})_n \otimes H_1(X, \mathbb{Z}) \otimes C$ where $C$ is equipped with the structure of module over $\mathbb{Z}[H_1(X, \mathbb{Z})]$ using $\rho$. If $\rho \neq 1$ then

$$H_1(\mathbb{X}, C) \otimes C[H_1(X, \mathbb{Z})] \otimes C = H_1(X, \rho). \quad (1.4.1.1)$$

This follows, for example, from the exact sequence of the low degree terms in the spectral sequence corresponding to the action of $H_1(X, \mathbb{Z})$ on the universal abelian cover $\mathbb{X} : H_p(H_1(X, \mathbb{Z}), H_q(\mathbb{X}, \rho)) \to H_{p+q}(X, \rho)$ (here $H_q(\mathbb{X}, \rho)$ denotes the homology of the complex $C_1(\mathbb{X})_n \otimes C$ with the action of $H_1(X, \mathbb{Z})$ changed by the character $\rho$, cf. Ch. XVI, Th. 8.4 in (Cartan and Eilenberg). This exact sequence is:

$$H_2(X, \rho) \to H_2(H_1(X, \mathbb{Z}), \rho) \to H_2(\mathbb{X}, \rho) \to H_1(X, \rho) \to H_1(H_1(X, \mathbb{Z}), \rho) = 0,$$

cf. Ch. XVI, (4a) in (Cartan and Eilenberg). For $\rho \neq 1$ we have $H_1(H_1(X, \mathbb{Z}), \rho) = 0$, so we obtain (1.4.1.1). For $\rho = 1$, an argument similar to sect. 1 in (Libgober, 1992) yields that $\dim H_1(\mathbb{X}, C) \otimes C[H_1(X, \mathbb{Z})] \otimes C$ is the dimension of the kernel of the map $\cup_X : H^*(\mathbb{X}, C) \to H^*(\mathbb{X}, C)$ given by the cup product. From the definition of Fitting ideals (cf. 1.2.1) it follows that for $\rho \neq 1$ one has:

$$\bigcap \{\rho \in \text{Hom}(G, C^*) | H_1(X, \rho) \bigcap \bigcirc \} \quad (1.4.1.2)$$

and that $\rho = 1$ belongs to $\text{Vim Ker}_{\cup_X}$, cf. Prop. 1.1.1 in (Libgober, 1992).

For example if $G = F_p$ then $\dim H_0(F_p, \rho)$ is 0, if $\rho$ is non trivial, and 1 otherwise. Using $e(F_p, \rho) = r - 1$ we obtain that $\dim H^1(F_p, \rho)$ is $r - 1$, if $\rho$ is non trivial, and otherwise is $r$. Since $\dim \ker_{\cup_F} = \frac{1}{2}$ we recover the description of the characteristic varieties for $F_p$ mentioned in (1.2.2.1).

1.4.2. Structure of characteristic varieties

We will need the following theorem of D. Arapura (1997) which generalizes the results of C. Simpson to quasi-projective case.

Let $X$ be a Kähler manifold with $H^1(X, C) = 0, D$ a normal crossings divisor and $X = X - D$. Then, for each characteristic variety $V$, there exist a finite number of torsion characters $\rho_j \in \text{Hom}(G, C^*)$, a finite number of unitary characters $\rho_j'$ and surjective maps onto (quasiprojective) curves $F_j : X \to C$ such that

$$V(X) = \bigcup_{\rho_j} H_1^1(C_i, C^*) \cup \bigcup_{\rho_j'}.$$

(1.4.2.1)
A consequence of 1.4.2.1 for curves in $C^2$ is that the components of positive dimensions of their characteristic varieties are subtori of $C^r$ translated by points of finite order.

1.4.3. Essential for a given set of components tori

By coordinate torus (corresponding to components $C_i$, $i = 1, \ldots, r$) we shall mean a subtorus of $C^r$ given by

$$t_i = \cdots = t_i = 1. \tag{1.4.3.1}$$

The inclusion $I_1, \ldots, I_r : C^2 - \bigcup_{i=1}^r C_i \to C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i$ induces a surjective map $I_1, \ldots, I_r : \pi_1(C^2 - \bigcup_{i=1}^r C_i) \to \pi_1(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i)$ with restriction $I_1, \ldots, I_r : \pi_1(C^2 - \bigcup_{i=1}^r C_i) \to \pi_1(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i)$ which is also surjective. Indeed if $K = \ker\pi_1(C^2 - \bigcup_{i=1}^r C_i) \to H_1(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i)$ then the map $K \to \pi_1(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i)$ is surjective. Since $K'$ is a normal closure of $\pi_1(C^2 - \bigcup_{i=1}^r C_i)$ and loops trivial in $\pi_1(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i)$ (e.g. loops which consist of paths from the base point to a point in vicinity of $C_i$, $i \neq i_1, \ldots, i_r$) and loops bounding small disk transversal to $C_i$ the surjectivity of $I_1, \ldots, I_r$ follows.

The latter gives rise to a surjective map $V_k(C^2 - \bigcup_{i=1, \ldots, r} C_i)\to V_k(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i)$ which induces an injection of corresponding characteristic varieties:

$$V_k(C^2 - \bigcup_{i=1, \ldots, r} C_i) \to V_k(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i) \tag{1.4.3.2}$$

(cf. Lemma 1.2.1). A component of $V_k(C^2 - \bigcup_{i=1, \ldots, r} C_i)$ which is an image of a component for some $i_1, \ldots, i_r$ in (1.4.3.2) is called obtained via a pull back. A component of $V_k(C^2 - \bigcup_{i=1, \ldots, r} C_i)$ is called essential if it is not a pull back of component of a characteristic variety of a curve composed of irreducible components of $C$.

**Lemma 1.4.3.** Let $V$ be a connected component of the characteristic variety $V_1$ of $C$ having positive dimension and belonging to the coordinate torus $t_i = \cdots = t_i = 1$. Then it is obtained via a pull back of a component of characteristic variety for the union of components of $\bigcup C_i$ ($i \neq i_1, \ldots, i_r$).

PROOF. According to Arapura Theorem (cf. (1.4.2) in (Arapura)) component $V$ defines a map $f : C^2 - C \to C$ for some quasi-projective curve $C$ such that for some local system $E \in \text{Char} \pi_1(C^2 - C)$ one has: $V = E \otimes f^*(\text{Char} C)$ where $\text{Char} C = \text{Hom}(\pi_1(C), C^r)$. We claim that $f$ factors as follows:

$$C^2 - C \xrightarrow{t_1 = \cdots = t_r} C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i \xrightarrow{f} C \tag{1.4.3.3}$$

The lemma is a consequence of existence of $f$. Indeed for almost all local systems $L$ on $C$ we have $H^1(E \otimes f^*(L)) = H^1(f_*E \otimes L)$ (cf. proof of Prop. 1.7 in (Arapura)). Moreover $H^1(f_*E \otimes L) = H^1(f_*((I_1, \ldots, i_r)_*)E \otimes f^*L)$ and the latter has the same dimension for almost all $L$ by the same argument from the proof of Prop. 1.7 in (Arapura).

To show the existence of $f$, let $D = C - C$ where $C$ is a non singular compactification of $C$. Since for $j = i_1, \ldots, i_r$ we have $t_j = 1$ on a translate (i.e. a coset of $f^*(\text{Hom}(H_1(C), C^r))$ and hence $t_j = 1$ on the latter subgroup of $\text{Char} (\pi_1(C^2 - C))$ we have for any $\chi \in \text{Char} (H_1(C, Z))$ and $j = i_1, \ldots, i_r$ the following: $\chi(f_*(\gamma_j)) = f^*(\chi)(\gamma_j) = t_j(f^*(\chi)) = 1$. Equivalently $f_*(\gamma_j) = 0$. Thus $f_*(\gamma_j) = 0$ in $H_1(C, Z)$ for $j \in (C^2 - C, Z)$ if and only if $j$ belongs to the subgroup generated by $\gamma_1, \ldots, \gamma_r$. Let us consider the pencil of curves on $\mathbb{P}^2$ formed by the fibres of $f$, $f$ extends to the map from the complement to the base locus of this pencil to $C^2$. Preimage of $D$ in this extension is a union of components of $C$ and we want to show that none of these components is $C_i$ with $i = i_1, \ldots, i_r$. But none of the components $C_i, i = i_1, \ldots, i_r$ is taken by this extension into $D$ since otherwise $f_*(\gamma_j) \neq 0$ for the corresponding $\gamma_j$. Hence domain of this extension of $f$ contains all points of $C_i \cup \cdots \cup C_i$ not belonging to the remaining components of $C$.

Note that it can occur that $H^1(C^2 - \bigcup_{i \neq i_1, \ldots, i_r} C_i, L) \neq H^1(C^2 - C, f^*(L))$ for isolated points of characteristic varieties, as is shown by examples in (Cohen and Suciu).

**1.5. ADJOINTS FOR COMPLETE INTERSECTIONS**

1.5.1. Let $F \subset \mathbb{P}^n$ be a surface which is a complete intersection given by the equations:

$$\tilde{F}_1 = \cdots = \tilde{F}_{n-2} = 0 \tag{1.5.1.1}$$

of degrees $d_1, \ldots, d_{n-2}$ respectively. Let (cf. p. 242 in (Hartshorne))

$$\Omega_{\tilde{F}} = \mathcal{E}xt^{n-2}(\mathcal{O}_F, \Omega_{\mathbb{P}^n})$$

\footnote{Incidentally, since the resolution $\tilde{X}$ of the base locus of this pencil is simply-connected one has $\tilde{C} = \mathbb{P}^1$.}
be the dualizing sheaf of \( F \). From the latter and the Koszul resolution

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(d_1 + \cdots + d_{n-2} - n - 1) \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(-d_1 - \cdots - d_{n-2}) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_F \rightarrow 0
\]  

(1.5.1.2)

it follows that one can identify \( \Omega_F \) with \( \mathcal{O}_F(d_1 + \cdots + d_{n-2} - n - 1) \).

Let \( f : \tilde{F} \rightarrow F \) be a resolution of singularities of \( F \) and \( \tau: f_*(\mathcal{O}_F) \rightarrow \mathcal{O}_F \) the trace map (Blass and Lipman). It identifies sections of \( f_*(\mathcal{O}_F) \) over an open set with those meromorphic differentials on non singular part of the open set in \( F \) that when pulled back on a resolution \( \tilde{F} \) admits a holomorphic extension over the exceptional set of \( F \). The adjoint ideal \( \mathcal{A}' \) is the annihilator of the cokernel of \( \tau \):

\[
\mathcal{A}' = \text{Hom}_{\mathcal{O}_F}(\mathcal{O}_F, f_*(\mathcal{O}_F)) = f_*(\mathcal{O}_F)(-d_1 - \cdots - d_{n-2} + n + 1) \tag{1.5.1.3}
\]

We define the sheaf of adjoint ideals on \( \mathbb{P}^n \) as \( \mathcal{A} = \pi^{-1}(\mathcal{A}') \) (also denoted as \( \text{Ad}_jF \)) where \( \pi \) is the most right map in (1.5.1.2). The degeneration of Leray spectral sequence for \( f \) (due to the Grauer-Riemenschneider vanishing theorem (Grauer and Riemenschneider, 1970)) yields

\[
H^j(\tilde{F}, \Omega_{\tilde{F}}) = H^j(F, \mathcal{A}_F) = H^j(F, \mathcal{A}')(d_1 + \cdots + d_{n-2} - n - 1) = H^j(\mathbb{P}^n, \mathcal{A}(d_1 + \cdots + d_{n-2} - n - 1))
\]  

(1.5.1.4)

In particular the irregularity of \( \tilde{F} \) i.e. \( \dim H^1(\tilde{F}, \mathcal{O}) \) can be found as the difference between the actual dimension \( H^0(\mathbb{P}^n, \mathcal{A}(d_1 + \cdots + d_{n-2} - n - 1)) \) and the “expected” dimension (i.e. \( \chi(\mathcal{A}(d_1 + \cdots + d_{n-2} - n - 1)) \)) of the adjoints (since \( H^i(\mathbb{P}^n, \mathcal{A}(d_1 + \cdots + d_{n-2} - n - 1)) = 0 \) for \( i \geq 2 \)).

1.5.2. Local description of adjoint ideals

Let

\[
F_i(w_1, \ldots, w_n) = 0, \ldots, F_{n-2}(w_1, \ldots, w_n) = 0 \tag{1.5.2.1}
\]

be a germ of a complete intersection of hypersurfaces in \( \mathbb{C}^n \) having an isolated singularity at the origin \( O \). For any two pairs \( 1 \leq i, j \leq n, i \neq j \) and \( 1 \leq k, l \leq n, k \neq l \) we have up to sign:

\[
\frac{dw_i \wedge dw_j}{\frac{\partial(F_1, \ldots, F_{n-2})}{\partial(w_1, \ldots, w_j, \ldots, w_n)}} = \frac{dw_k \wedge dw_l}{\frac{\partial(F_1, \ldots, F_{n-2})}{\partial(w_1, \ldots, w_2, \ldots, w_{k}, \ldots, w_n)}} \tag{1.5.2.2}
\]

Indeed the Cramer’s rule for the solutions of the system of equations:

\[
\frac{\partial F_k}{\partial w_1} dw_1 \wedge dw_i + \cdots + \frac{\partial F_k}{\partial w_n} dw_n \wedge dw_i = 0 \quad (k = 1, \ldots, n - 2)
\]

when one views \( dw_k \wedge dw_i (k = 1, \ldots, n - 1) \) as unknowns yields that up to sign:

\[
dw_k \wedge dw_i = \frac{\frac{\partial(F_1, \ldots, F_{n-2})}{\partial(w_1, \ldots, w_i, \ldots, w_n)}}{\frac{\partial(F_1, \ldots, F_{n-2})}{\partial(w_1, \ldots, w_2, \ldots, w_{k}, \ldots, w_n)}} \tag{1.5.2.3}
\]

(1.5.2.2) follows from this for any two pairs \((i, j), (k, l), i \neq j, k \neq l \).

Since \( F_1 = \cdots = F_{n-2} = 0 \) is a complete intersection with isolated singularity one of the Jacobians \( \frac{\partial(F_1, \ldots, F_{n-2})}{\partial(w_1, \ldots, w_i, \ldots, w_n)} \) is non vanishing in a neighborhood of the singularity everywhere except for the singularity itself. In particular each side (1.5.2.2) defines a holomorphic 2-form outside of the origin for any \((i, j), i \neq j \) or \((k, l), k \neq l \). In fact this form is just the residue of the log-form \( dw_1 \wedge \cdots \wedge dw_{n-1} \) at non singular points (i.e. outside of the origin of (1.5.1.1)).

The adjoint ideal \( \mathcal{A}_O \) in the local ring \( \mathcal{O}_O \) of the origin of a germ of complete intersection (1.5.2.1), according to the description of the trace map 1.5.1 can be made explicit as follows. Let \( f : \mathbb{C}^n \rightarrow \mathbb{C}^n \) be an embedded resolution of (1.5.1.1). Then \( \mathcal{A}_O \) consists of \( \phi \in \mathcal{O}_O \) such that \( f^{-1}(\phi) = \frac{dw_1 \wedge \cdots \wedge dw_n}{\frac{\partial(F_1, \ldots, F_{n-2})}{\partial(w_1, \ldots, w_i, \ldots, w_n)}} \) admits a holomorphic extension from \( f^{-1}(\mathbb{C}^n - O) \) to \( \mathbb{C}^n \).

Similarly, the elements of \( H^0(\mathcal{A}(d_1 + \cdots + d_{n-2} - n - 1)) \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d_1 + \cdots + d_{n-2} - n - 1)) \) can be viewed as meromorphic forms with log singularities near non singular points of (1.5.1.1) having as residue a 2-form on a non singular locus of \( F \) and admitting a holomorphic extension on \( \tilde{F} \).

2. Ideals and polytopes of quasiaugmentation

2.1. IDEALS OF QUASIAUGMENTATION

Let \( f \) be a germ of a reduced algebraic curve having a singularity with \( r \) irreducible branches at the origin of \( \mathbb{C}^2 \) near which it is given by local equation \( f = f_1(x, y) \cdots f_r(x, y) = 0 \). Let \( O \) be the local ring of the origin and \( A \) be an ideal in \( O \).

**Definition 2.1.1.** An ideal \( A \) is called an ideal of quasiaugmentation of \( f \) with parameters \((f_1, \ldots, f_r, m_1, \ldots, m_r) \) (\( f_i, m_i \) are integers) if \( A = \{ \phi \in O \mid z_1^{m_1} \cdots z_r^{m_r} \phi \in \text{Ad}_jV_{(m_1, f_1), \ldots, (m_r, f_r)} \} \) where \( V_{(m_1, f_1), \ldots, (m_r, f_r)} \) is a germ at the origin of the complete intersection in \( \mathbb{C}^{*+2} \) given by the equations:

\[
z_1^{m_1} = f_1(x, y), \ldots, z_r^{m_r} = f_r(x, y). \tag{2.1.1}
\]

An ideal of quasiaugmentation is an ideal in \( O \) which is an ideal of quasiaugmentation for some system of parameters.
2.2. BASIC IDEAL

Let \( A(f_1, \ldots, f_r) \subset \mathcal{O} \) be the ideal generated by

\[
\begin{align*}
\frac{(f_1)^i}{f_1} f_1 f_2 \cdots f_r & \quad (f_2)^i_j/f_2 f_1 f_2 \cdots f_r, \quad (i = 1, \ldots, r), \\
\text{Jac}(\frac{(f_i f_j)}{f_i f_j}) & \quad f_i f_1 \cdots f_r, \quad (i, j = 1, \ldots, r, i \neq j)
\end{align*}
\]

(we shall call it the basic ideal).

Equating all polynomials (2.2.1) to zero yields a system of equations having (0, 0) as the only solution. Therefore \( \mathcal{O}/A(f_1, \ldots, f_r) \) is an Artinian algebra.

Moreover for any set of parameters \((i_1, \ldots, i_r; m_1, \ldots, m_r)\) the corresponding ideal of quasiadjunction contains \( A(f_1, \ldots, f_r) \). Indeed, if \( F_i = z_i^{m_i - f_i(x,y)} \), then up to sign

\[
\frac{\partial (f_1 \cdots f_r)}{\partial (z_1 \cdots z_r)} = z_1^{m_1 - 1} \cdots z_i^{m_i - 1} \cdots z_r^{m_r - 1}, \quad (f_i)_x \quad \text{and hence}
\]

\[
\frac{(f_1)_x f_1 \cdots f_i f_i \cdots f_r dz_i \wedge dy}{\partial (f_1 \cdots f_r)} = \frac{(f_1)_x f_1 \cdots f_i f_i \cdots f_r dx \wedge dz_i}{z_1^{m_1 - 1} \cdots z_i^{m_i - 1} \cdots z_r^{m_r - 1}(f_i)_x} = z_1 \cdots z_i \cdots z_r dz_i \wedge dy
\]

which is holomorphic on \( \mathbb{C}^{r+2} \). Similarly, one sees that the 2-forms corresponding to other generators of \( A(f_1, \ldots, f_r) \) coincide on \( F_1 = \cdots = F_r = 0 \) with the forms admitting a holomorphic extension to \( \mathbb{C}^{r+2} \).

In particular, there are only finitely many ideals of quasiadjunction.

2.3. IDEALS OF QUASIADJUNCTION AND POLYTOPES

Let \( \mathcal{U} = \{(x_1, \ldots, x_r) \in \mathbb{P}^r | 0 \leq x_i < 1\} \) be the unit cube with coordinates corresponding to the components of a curve \( C \). Sometimes we shall denote this cube as \( \mathcal{U}(C) \). If \( C \) is formed by components of \( C \) then we shall view \( \mathcal{U}(C) \) as the face of \( \mathcal{U}(C) \) given by \( x_j = 0 \) where \( j \) runs through indices corresponding to components of \( C \) not belonging to \( C \).

By a **convex polytope** mean a subset of \( \mathbb{R}^m \) which is the convex hull of a finite set of points with some faces possibly deleted. By a **polytope** we mean a finite union of convex polytopes. Class of polytopes in this sense is closed under finite unions and intersections. A complement to a polytope within an ambient polytope is a polytope. By **face of maximal dimension** of a polytope we mean the intersection of the polytope's boundary with the hyperplane for which this intersection has the dimension equal to the dimension of the boundary. **A face of**

---

\(^3\) Only subsets of \( \mathcal{U} \) will occur below.
For given \( k \) let \( f_k(\phi) \) be the minimal integer solution with \( f_k(\phi) \) considered as unknown, for this inequality and \( \phi_k \) be such that \( f_k(\phi_k) = f_k(\phi) \). In other words \( \phi \in A \) if and only if \( f_k(\phi) \geq f_k(A) \). We have \( f_k(A) = \left\lfloor \sum_{i=1}^r \left( a_{k_i} - (j_i + 1)ak_i/m_i \right) \right\rfloor \) where \( \{r\} \) resp. \( \{r\} \) denotes the smallest integer that is strictly greater than (resp. the integer part of) \( r \). We shall call \( f_k(A) \) the multiplicity of \( A \) along \( E_k \). This is the minimum of multiplicities along \( E_k \) of pull backs on \( Y_f \) of elements of \( A \).

The same calculation shows that \( z_i^{m_i} \cdots z_r^{m_r} \phi \), where \( \phi \) belongs to an ideal of quasiadjunction \( A \), is in the adjoint ideal of \( z_i^{m_i} = f_i(x,y), \ldots, z_r^{m_r} = f_r(x,y) \) if and only if:

\[
\sum_{i=1}^r (j_i - m_i + 1) \frac{m_i \cdots m_r \cdot a_{k_i}}{g_1 \cdots g_r \cdot s_k} + \sum_{i=1}^r (j_i - m_i) \frac{m_i \cdots m_r}{g_1 \cdots g_r \cdot s_k} - 1 \geq 0
\]

To see (2.3.3), we can select local coordinates \( (u, v) \) on \( Y_f \) near a point belonging to a single component \( E_k \) in which the latter is given by the equation \( u = 0 \). Then \( \rho^*(\gamma_i(x,y)) = u^{a_{k_i}} \cdot e_i(u,v) \) where \( e_i(u,v) \) are units in the corresponding local ring. The fiber product \( Y_f \times_{\gamma_i} V(m_1,\ldots,m_r) \) is a subvariety in \( \mathbb{C}^{r+2} \times_{\gamma_i} Y_f \) given by the equations:

\[
z_i^{m_i} = u^{a_{k_i}} e_i(u,v), \ldots, z_r^{m_r} = u^{a_{k_r}} e_r(u,v)
\]

Each branch of (2.3.4) has the following local parameterization:

\[
u = \frac{x_i^{j_i - m_i} - x_i^{j_i - m_i}}{x_i^{j_i - m_i} - x_i^{j_i - m_i}}, \quad z_i = \frac{x_i^{1} - x_i^{m_i}}{x_i^{1} - x_i^{m_i}}, \quad i = 1, \ldots, r
\]

(exponents are chosen so that their greatest common divisor will be equal to 1 and so that they will satisfy (2.3.4)). This yields the first equality in (2.3.3). We have

\[
\rho^*(\gamma_i(x,y)) = \sum_{i=1}^r (j_i - m_i + 1) \frac{m_i \cdots m_r \cdot a_{k_i}}{g_1 \cdots g_r \cdot s_k} + \sum_{i=1}^r (j_i - m_i) \frac{m_i \cdots m_r}{g_1 \cdots g_r \cdot s_k} - 1 \geq 0
\]

Hence the second equality in (2.3.3) follows from (2.3.5).

Finally, since the map \( \pi^*(\gamma_i(x,y)) \) is given locally by \( (t, v) \rightarrow (t^{x_i^{j_i - m_i}}, v) \), we have:

\[
\rho^*(\gamma_i(x,y)) (dx \wedge dy) = \pi^*(\gamma_i(x,y)) (dx \wedge dy) (t^{x_i^{j_i - m_i}} \cdot v^2 dt \wedge dv)
\]

which implies the last equality in (2.3.3).

Now it follows from (2.3.2) and (2.3.5) that \( \phi \in A(j_1, \ldots, j_r|m_1, \ldots, m_r) \) if and only if for any \( k \) the multiplicity \( f_k(\phi) \) satisfies:

\[
\sum_{i=1}^r (j_i - m_i + 1) \frac{m_i \cdots m_r \cdot a_{k_i}}{g_1 \cdots g_r \cdot s_k} + \sum_{i=1}^r (j_i - m_i) \frac{m_i \cdots m_r}{g_1 \cdots g_r \cdot s_k} - 1 \geq 0
\]

2.4. LOCAL POLYTOPES OF QUASIADJUNCTION AND FACES OF QUASIADJUNCTION

**Definition 2.4.1.** We say that two points in the unit cube \( U \) are equivalent if the collections of polytopes \( \tilde{A}(A) \) containing each of the points coincide. A
(local) polytope of quasiadjunction $\Delta$ is an equivalence class of points with this equivalence relation.

**Definition 2.4.2.** A face of quasiadjunction is an intersection of a face (cf. (2.3)) of a local polytope of quasiadjunction and a (different) polytope of quasiadjunction. In particular, each face of quasiadjunction belongs to a unique polytope of quasiadjunction.

For each face let us consider the system of equations defining the affine sub-space of $Q'$ spanned by this face. One can normalize the system so that all coefficients of variables and the free term are integers and the g.c.d. of non zero minors of maximal order is equal to 1.

**Definition 2.4.3.** The order of a face of quasiadjunction is the g.c.d of minors of maximal order in the matrix of coefficients in the normalized system of linear equations defining this face.

This is the order of the torsion of the quotient of $\mathbb{Z}^r$ by the subgroup generated by the vectors having as coordinates the coefficients of variables in the equations of the face. In particular this integer is independent of the chosen normalized system of equations and depends only on the face of quasiadjunction.

The (local) ideal of quasiadjunction corresponding to a face of quasiadjunction is the ideal of quasiadjunction corresponding to the polytope of quasiadjunction containing this face. The ideal corresponding to a face of quasiadjunction has the form $A(\{j_1, \ldots, j_r\}, \{m_1, \ldots, m_r\})$ where $\{\frac{j_1+1}{m_1}, \ldots, \frac{j_r+1}{m_r}\}$ belongs to this face.

2.5. **EXAMPLES**

1. In the case of the branch with one component the ideals of quasiadjunction correspond to the constants of quasiadjunction, cf. (Libgober, 1983). Recall that for a germ $\phi$ the rational number $\kappa_{\phi}$ is characterized by the property $\min\{|i|z^i\phi \in \text{Ad}(j^2 = f(x,y))\} = [\kappa_{\phi}, n]$. The ideal of quasiadjunction corresponding to $\kappa$ consists of $\phi$ such that $\kappa_{\phi} > \kappa$. For example for the cusp $x^2 + y^3$ the only non zero constant of quasiadjunction is 1/6, there are two polytopes of quasiadjunction i.e. $\Delta' = \{x \in [0,1], 1 > x > 1/6\}$ and $\Delta'' = \{x \in [0,1], 1/6 > x > 0\}$. $x = 1/6$ is the face of quasiadjunction and the corresponding ideal of quasiadjunction is the maximal ideal. For an arbitrary unbranched singularity we have $\Delta(A_k) = \{x | x > \kappa\}$. The order of the face $x = \kappa$ is equal to the order of the root of unity $\exp(2\pi i/\kappa)$.

2. Let us consider a tacnode, locally given by the equation: $y(y - x^2) = y_g(x,y) = y - x^2$. Then the basic ideal is the maximal ideal and hence there is only one ideal of quasiadjunction. If $\mathcal{M}$ is the maximal ideal, then $\Delta(M)$ is the whole unit square. To determine the polytope $\Delta(\mathcal{O})$, note that after two blow-ups we obtain an embedded resolution which in one of the charts looks like: $x = u v, y = u^2 v$ where $u = 0$ and $v = 0$ are the exceptional curves. Hence for the component $u = 0$ we obtain $a = b = 2, f(\phi = 1) = 0, c = 2$ and the corresponding polytope is $x+y = 1/2$. The face of quasiadjunction is $x+y = 1/2$ and the corresponding ideal is the maximal one.

3. For the ordinary singularity of multiplicity $m$: $(\alpha_1 x + \beta_1 y) \cdots (\alpha_m x + \beta_m y) = 0$ the basic ideal is $\mathcal{M}_{m-2}$ where $\mathcal{M} \subset \mathcal{O}$ is the maximal ideal of the local ring at the origin. Since the resolution can be obtained by a single blow up, we have $a_i = 1, c_i = 1, i = 1, \ldots, m$ i.e. the polytope $\Delta(A)$ of an ideal of quasiadjunction $A$ is:

$$x_1 + \cdots + x_m > m - 2 - f_1(A)$$

Since $f_1(\phi) \geq f_1(A)$ is equivalent to $\phi \in \mathcal{M}^{f_1(A)}$ i.e. the latter is the ideal corresponding to the polytope (2.5.1). The faces of quasiadjunction are $x_1 + \cdots + x_m = m - 2 - f_1(A)$ ($f_1(A) = 0, \ldots, m - 3$) and the corresponding ideal of quasiadjunction is $\mathcal{M}^{f_1(A)+1}$.

Additional examples are discussed in (Libgober, 2001).

2.6. **GLOBAL POLYTOPES AND SHEAVES OF QUASIADJUNCTION**

Let $\mathbb{R}^r$, as in 2.3, be vector space coordinates of which are in one to one correspondence with the components of the curve $C = \bigcup_{i=1}^{m} C_i$. For a singular point $p$ of $C$, let $C_p$ be the collection of components of $C$ passing through $p$. Each polytope of quasiadjunction $\Delta_p \subset \mathcal{U}(C_p)$ of $p$ defines the polytope in $\mathcal{U} = \{(x_1, \ldots, x_r) | 0 \leq x_i \leq 1\} \subset \mathbb{R}^r$ consisting of points $(x_1, \ldots, x_r) \in \mathcal{U}$, $(x_1, \ldots, x_{i_p}) \in \Delta_p$ where $(i_1, \ldots, i_{r(p)})$ are the coordinates corresponding to the components of $C_p$ (i.e. passing through the singularity $p$). We shall use the notation $\Delta_p(\mathcal{U})$ for this polytope in $\mathcal{U}$.

2.6.1. **Definition**

A type of a point in $\mathcal{U}$ is the collection of polytopes $\Delta_p(\mathcal{U}) \subset \mathcal{U}$ to which this point belongs, where $p$ runs through all $p \in \text{Sing} C$.

We call two points in $\mathcal{U}$ equivalent if they have the same type. A (global) polytope of quasiadjunction is an equivalence class of this equivalence relation. Global polytopes of quasiadjunction form a partition which is a refinement of every partitions of $\mathcal{U}$ defined by polytopes $\Delta_p(\mathcal{U})$ corresponding to local polytopes of quasiadjunction of singularities of $C$.

2.6.2. **Definition**

We shall call a face $\delta$ of quasiadjunction contributing if it belongs to a hyperplane $d_1 x_1 + \cdots + d_r x_r = l$ where $d_1, \ldots, d_r$ are the degrees of the components of $C$ corresponding to respective coordinates $x_1, \ldots, x_r$. This hyperplane is called contributing, the integer $l = l(\delta)$ is called the level of both the contributing hyperplane and contributing face. The order of a global face of quasiadjunction is defined as
in local case (cf. 2.4.2). A polytope of quasidejunction is called contributing if it contains a contributing face.

A point \((x_1, \ldots, x_r) \in \delta\) is called interior \(C\)-point for a curve \(C\) formed by components of \(C\) if \(x_i \not= 1\) if and only if \(i\) corresponds to a component of \(C\) and \((x_1, \ldots, x_r)\) is in the interior of \(\delta\).

**Remarks 2.6.2.1.** In the case when \(r = 1\), e.g. for an irreducible curve, an order of a global face of quasidejunction is the order of a root of the Alexander polynomial. Indeed, for a constant of quasidejunction \(\kappa\), \(\exp(2\pi i \kappa)\) is a root of Alexander polynomial (Loeser and Vaquie).

2.6.2.2. The collection of orders of faces of quasidejunction for reducible curves a priori cannot be determined just by the local types of all singular points. However it is combinatorial invariant of the curve in the sense that it depends only on local information about singularities and specification of components which contain specified singular points, e.g. it is independent of the geometry of the set of singular points in \(\mathbb{P}^2\). In the case of arrangements of lines this is combinatorial invariant in the common sense of the word.

### 2.6.3. Definition

The sheaf of ideals \(\mathcal{A}(\delta) \subset O_{\mathbb{P}^2}\) such that \(\text{Supp}(O_{\mathbb{P}^2}/\mathcal{A}) \subset \text{Sing} C\) is called the sheaf of ideals of quasidejunction corresponding to the face of quasidejunction \(\delta\) if the stalk \(\mathcal{A}_p\) at each singular point \(p \in C\) with local ring \(O_p\) is the ideal \(A\) of quasidejunction corresponding to the face \(\Delta_p = \Delta \cap H_p\) with \(H_p \subset \mathbb{R}^2\) being given by \(x_i = 0\) where \(i\) are the coordinates corresponding to the components of \(C\) not passing through \(p\).

### 2.6.4. Examples

1. (Libgober, 1983) For an irreducible curve of degree \(d\) with nodes and the ordinary cusps as the only singularities the global polytope of quasidejunction coincide with the local one of the cusp. The only face of quasidejunction is \(x = 1/6\). The contributing hyperplane is given by \(dx = d/6\) and its level is \(d/6\). The sheaf of quasidejunction corresponding to this face of quasidejunction is the ideal sheaf having stalks different from the local ring only at the points of \(\mathbb{P}^2\) where the curve has cusps and the stalks at those points are the maximal ideals of the corresponding local rings.

2. Let us consider \(C\) which is an arrangement of lines. For a point \(P\) let \(m_P\) denotes the multiplicity. We consider only points with \(m_P > 2\). Each global face of quasidejunction is a solution of a system of equations:

\[
L_P : \quad x_1 + \cdots + x_m = s_P
\]

where \(s_P = 1, \ldots, m_P - 2\) (cf. example 3 in 2.5). The indices of variables \(x\) correspond to the lines of the arrangement and \(x_i\) appears in \(L_P\) if and only if it correspond to a line passing through \(P\). Each system (2.6.1) corresponding to a face of quasidejunction singles out a collection of vertices of the arrangement. This face is contributing if the equation

\[
x_1 + \cdots + x_r = k, \quad (k \in \mathbb{N})
\]

is a linear combination of equations (2.6.1). The level of such contributing face is \(k\). Its order is the g.c.d of minors of maximal order in system (2.6.1).

### 3. The first Betti number of an abelian cover

In this section we shall prove a formula for the irregularity of an abelian covers of \(\mathbb{P}^2\) branched over \(C\) in terms of the polytopes of quasidejunction introduced in the last section. More precisely we shall calculate the multiplicity of a character of the Galois group of the cover acting on the space \(H^{1,0}(\hat{P}_{m_1, \ldots, m_r})\) of holomorphic \(1\)-forms. We translate this into an information about characteristic varieties of the fundamental group and consider several examples of characteristic varieties for the fundamental groups of the complements to arrangements of lines.

#### 3.1. STATEMENT OF THE THEOREM

Let \(C = \bigcup_{i=1}^{r} C_i\) be a reduced curve \(f(u,x,y) = f_1(u,x,y) \cdots f_r(u,x,y)\) with \(r\) irreducible components and the degrees of components equal to \(d_1, \ldots, d_r\) and \(d = d_1 + \cdots + d_r\) be the total degree of \(f(u,x,y) = 0\). Let \(L_\infty\) be the line \(u = 0\) at infinity which, as above, we shall assume transversal to \(C\) (cf. 1.1).

a) The irregularity of a desingularization \(\hat{P}_{m_1, \ldots, m_r}\) of an abelian cover of \(\mathbb{P}^2\) branched over \(C \cup L_\infty\) and corresponding to the surjection \(\pi_1(\mathbb{P}^2 - C \cup L_\infty) \to \mathbb{Z}/m_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r \mathbb{Z}\) is equal to

\[
\sum_{C'} \left( \sum_{\delta(C')} N(\delta(C')) \cdot \dim H^1(\mathcal{A}_{\delta(C')}(d - l(\delta(C)) - 3)) \right)
\]

where the summations are over all curves \(C'\) formed by the components of \(C\) and the contributing faces of quasidejunction \(\delta(C')\) respectively. Here \(l(\delta(C'))\) is the level of the contributing face of \(\delta(C')\) and \(N(\delta(C'))\) is the number of interior \(C\)-points \((\frac{i_1+1}{m_1}, \ldots, \frac{i_r+1}{m_r})\) in the contributing face of \(\delta(C')\).

b) Let \(\chi\) be the character of \(\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}\) taking on \((a_1, \ldots, a_j, \ldots, a_r)\) value \(\exp(2\pi i \frac{a_j}{m_j})\). For a character \(\chi\) of \(\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}\), let

\[
H^1_{\chi}(\hat{P}_{m_1, \ldots, m_r}) = \{ x \in H^1(\hat{P}_{m_1, \ldots, m_r}), g \in \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}, g \cdot x = \chi(g) \cdot x \}.
\]

If \((\frac{i_1+1}{m_1}, \ldots, \frac{i_r+1}{m_r})\) is an interior \(C\)-point (cf. 2.3.2) belonging to the contributing face \(\delta\) then

\[
\dim H^1(\chi; \hat{P}_{m_1, \ldots, m_r}) = \dim H^1(\mathcal{A}_{\delta}(d - 3 - l(\delta)))
\]
c) Let \( t_l = \exp(2\pi \sqrt{-1} x_l) \). For each contributing face \( \delta \) belonging to \( U^\alpha \) and its image \( \bar{\delta} \) under the conjugation map (cf. 2.3), let \( L_\alpha(x_1, \ldots, x_r) = \bar{\beta}_\alpha \) be the system of equations defining it where \( L_\alpha(x_1, \ldots, x_r) \) is a linear form with integer coefficients such that g.c.d. of the minors of maximal order in the matrix of coefficients is equal to 1. Then the corresponding essential component of the characteristic variety of \( \pi_t(|P^2 - C \cup L_{\infty}) \), which either has a positive dimension or is a torsion point, is the intersection of cosets given by the equations:

\[
\exp(2\pi \sqrt{-1} L_\alpha) = \exp(2\pi \sqrt{-1} \bar{\beta}_\alpha)
\]  

(3.1.3)

written in terms of \( t_l \)'s. Vice versa, any essential component can be obtained in such way.

Note that c) implies that the essential components of the characteristic varieties are Zariski's closures of the images of the contributing faces under the exponential map. Indeed, since g.c.d. of minors of coefficients in \( L_\alpha \) is 1 the intersection of subgroups \( \exp(2\pi \sqrt{-1} L_\alpha) = 1 \) is connected and the closure of the image of the face of quasiadjunction is Zariski dense in the translation of this connected component given by (3.1.3).

3.2. PROOF OF THE THEOREM

We shall start with the case when \( m_i \geq d_i \) for \( i = 1, \ldots, r \). Let \( A \subset O_{P^{r+2}} \) be the sheaf of adjoint ideals of the complete intersection \( V_{m_1, \ldots, m_r} \subset P^{r+2} \) given by the equations (cf. (3.1.1.1)):

\[
z_{m_1}^{t_{m_1}} = u^{m_1 - d_1} f_1(u, x, y), \ldots, z_{m_r}^{t_{m_r}} = u^{m_r - d_r} f_r(u, x, y)
\]

(3.2.1)

\( V_{m_1, \ldots, m_r} \) provides a model of an abelian branched cover of \( P^2 \) branched over \( f_1 \cdot \cdots \cdot f_r = 0 \) and the line at infinity. \( V_{m_1, \ldots, m_r} \), has isolated singularities at the points of (3.2.1) which are above the singularities of \( C \) in \( P^2 - L_{\infty} \). The action of the Galois group of the cover is induced from the action of the product of groups of roots of unity \( \mu_{m_1} \times \cdots \times \mu_{m_r} \) on the \( P^{r+2} \) via multiplication of corresponding z-coordinates.

Let \( H \) be the set of common zeros of \( z_1, \ldots, z_r \in H^0(P^{r+2}, O(1)) \) and \( A_{t_1, \ldots, t_r} \) the subsheaf of \( O_{P^{r+2}} \) germs of section product of which with \( z_1^{t_1} \cdots z_r^{t_r} \) belongs to \( A \). The action of \( \mu_{m_1} \times \cdots \times \mu_{m_r} \) on \( P^{r+2} \) induces the action on \( A_{t_1, \ldots, t_r} \).

Let \( J_H \) be the ideal sheaf of the plane \( H \subset P^2 \). We have the following \( \mu_{m_1} \times \cdots \times \mu_{m_r} \)-equivariant sequence:

\[
0 \to A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3) \otimes J_H \to A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3) \to A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3)|_H \to 0
\]

(3.2.2)

Let

\[
F(i_1, \ldots, i_r) = \dim H^1(A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3)),
\]

\[
F_{\chi}(i_1, \ldots, i_r) = \dim \{ x \in H^1(A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3) | g : x = \chi(g)x, \forall g \in \mu_{m_1} \times \cdots \times \mu_{m_r} \}
\]

(3.2.3)

In particular \( F(0, \ldots, 0) \) is the irregularity of a nonsingular model of \( V_{m_1, \ldots, m_r} \).

Step 1. Degree of the curves in the linear system \( H^0(A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3)|H) \). Let us calculate the multiplicity of the line \( L_{\infty} : z_1 = \cdots = z_r = u = 0 \) as the fixed component of the curves in the linear system cut on \( H \) by the hypersurfaces in the linear system \( H^0(A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3)) \). This multiplicity is the smallest \( k \) such that \( u^k \) belongs to the latter system of hypersurfaces. In appropriate coordinates \( (z_1, \ldots, z_r, u, v) \) at a point \( P \) of this line outside of \( L_{\infty} \cap C \) (i.e. we have \( f_1(P) \cdots f_r(P) \neq 0 \)) the local equation of \( V_{m_1, \ldots, m_r} \) is \( z_1^{m_1} = u^{m_1 - d_1}, \ldots, z_r^{m_r} = u^{m_r - d_r} \). Let

\[
l = l \cdot c.m. \left( \frac{m_1}{m_1 - d_1} (m_1 - d_1) \cdots (m_r - d_r), \ldots, \frac{m_r}{m_r - d_r} (m_1 - d_1) \cdots (m_r - d_r), (m_1 - d_1) \cdots (m_r - d_r) \right)
\]

Then each branch of the normalization of \( V_{m_1, \ldots, m_r} \) has the parameterization \( (t, v) \) such that:

\[
z_1 = t^{m_1(m_1-d_1)/(m_1-d_1)-1}, \ldots, z_r = t^{m_r(m_r-d_r)/(m_r-d_r)-1}, u = t^{m_1-d_1}(m_1-d_1).
\]

Therefore the pull back of the form \( \frac{d_1}{z_1^{m_1-1}|z_1^{m_1-1}} \cdots \frac{d_r}{z_r^{m_r-1}|z_r^{m_r-1}} \) of \( V_{m_1, \ldots, m_r} \) to the \((t, v)\) chart is regular if and only if

\[
k > \sum_{j=1}^r (m_j - d_j - (i_j + 1)) + \frac{d_j(i_j + 1)}{m_j} - 1.
\]

(3.2.4)

The smallest \( k \) which satisfies this inequality, i.e. the multiplicity of the line \( u = 0 \) as the component of a generic curve from \( H^0(A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3)|H) \), is equal to

\[
\sum_j (m_j - d_j - (i_j + 1)) + \left\lfloor \frac{d_j(i_j + 1)}{m_j} \right\rfloor.
\]

(3.2.5)

As a consequence of this, the degree of the moving curves in the linear system \( H^0(A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3)|H) = \sum_j d_j - 3 - \left\lfloor \frac{d_j(i_j + 1)}{m_j} \right\rfloor \) and therefore the moving curves belong to the linear system \( H^0(A_{t_1, \ldots, t_r}((m_1 - i_1) + \cdots + (m_r - i_r) - r - 3)|H) \).

\[
\Delta = \text{polytope of quasiadjunction containing } (i_1 + 1/m_1, \ldots, i_r + 1/m_r).
\]
In fact the moving curves form a complete system since the cone over any curve in $H^0(\mathcal{A}(\Sigma j \langle d \rangle - 3 - [\Sigma j \langle d \rangle \langle (j+1) \rangle \langle m_j \rangle])$ belongs to $H^0(\mathcal{A}(\Sigma j \langle d \rangle - 3 - [\Sigma j \langle d \rangle \langle (j+1) \rangle \langle m_j \rangle])$).

**Step 2. A recurrence relation for $F(i_1, \ldots, i_r)$ and $F_X(i_1, \ldots, i_r)$.**

Let $s(i_1, \ldots, i_r) = \dim H^1(\mathcal{A}(\Sigma j \langle d \rangle - 3 - [\Sigma j \langle d \rangle \langle (j+1) \rangle \langle m_j \rangle])$ where $\Delta$ is the polytope of quasijunction of $C$ containing $(l+1 \langle m_1 \rangle, \ldots, l+1 \langle m_r \rangle)$ and $\varepsilon_X(i_1, \ldots, i_r) = 1$ (resp. 0) if $\chi = \chi_1^{l_1-1} \ldots \chi_r^{l_r-1}$ (resp. otherwise). We claim the following recurrence:

$$F(i_1, \ldots, i_r) = s(i_1, \ldots, i_r) +$$

$$+ \sum_{l=1}^{r} (-1)^{l+1} \sum_{i_j \in \leq l_j} F(\ldots, i_j + 1, \ldots, i_j + 1, \ldots) ;$$

$$F_X(i_1, \ldots, i_r) = \varepsilon_X(i_1, \ldots, i_r)s(i_1, \ldots, i_r) +$$

$$+ \sum_{l=1}^{r} (-1)^{l+1} \sum_{i_j \in \leq l_j} F_X(\chi_1^{l_1-1} \ldots \chi_r^{l_r-1}, \ldots, i_j + 1, \ldots) .$$

(3.2.6)

Equivalently the first of equalities (3.2.6) can be written as

$$s(i_1, \ldots, i_r) = \sum_{l=0}^{r} (-1)^l \sum_{i_j \in \leq l_j} F(\ldots, i_j + 1, \ldots, i_j + 1, \ldots)$$

and similarly for the second. This identity will be derived from the following. For $h$ such that $1 \leq h \leq r$ let

$$F(i_1, \ldots, i_r| q_1, \ldots, q_h) =$$

$$= \dim H^1(\mathcal{A}(i_1, \ldots, i_r)((m_1 - i_1) + \ldots + (m_r - i_r) - r - 3)|H_{q_1} \cap \cdots \cap H_{q_h})$$

where $H_i$ is the hyperplane $z_i = 0$ in $\mathbb{P}^{r+2}$ while for $h = 0$ we let $F(i_1, \ldots, i_r| q_0) = F(i_1, \ldots, i_r)$. In particular $s(i_1, \ldots, i_r) = F(i_1, \ldots, i_r| 1, \ldots, r)$. Similarly one defines $F_X(i_1, \ldots, i_r| q_1, \ldots, q_h)$. We shall prove by induction over $h$:

$$F(i_1, \ldots, i_r| q_1, \ldots, q_h) = \sum_{l=0}^{h} (-1)^l \sum_{i_j \in \leq l_j} F(\ldots, i_j + 1, \ldots, i_j + 1, \ldots) ;$$

$$F_X(i_1, \ldots, i_r| q_1, \ldots, q_h) =$$

$$= \sum_{l=0}^{h} (-1)^l \sum_{i_j \in \leq l_j} F_X(\chi_1^{l_1-1} \ldots \chi_r^{l_r-1}, \ldots, i_j + 1, \ldots) .$$

(3.2.7)

The identity (3.2.6) is a special case of (3.2.7) when $q_i = i$. For any $(i_1, \ldots, i_r | q_1, \ldots, q_h), (h \geq 0)$ from the exact sequence (in which the left map is the

**multiplication by $z_{q_h}$):**

$$0 \to \mathcal{A}(q_{h+1}) \to \mathcal{A}(q_{h+1} + i_1 + \ldots + (m_{q_{h+1}} - i_{q_{h+1}} - 1) + \ldots + (m_1 - i_1)) \to$$

$$\to \mathcal{A}((m_1 - i_1) + \ldots + (m_r - i_r) - r - 3)|H_{q_1} \cap \cdots \cap H_{q_h} \to$$

$$\to \mathcal{A}(q_{h+1} + i_1 + \ldots + (m_r - i_r) - r - 3)|H_{q_1} \cap \cdots \cap H_{q_{h+1}} \to 0$$

we obtain

$$F(i_1, \ldots, i_r| q_1, \ldots, q_h, q_h+1) = F(i_1, \ldots, i_r| q_1, \ldots, q_h) +$$

$$+ F(i_1, \ldots, i_r| q_{h+1} + 1, \ldots, i_r| q_1, \ldots, q_h)$$

(3.2.9)

$$F_X(i_1, \ldots, i_r| q_1, \ldots, q_h, q_h+1) = F_X(i_1, \ldots, i_r| q_1, \ldots, q_h) +$$

$$- F_X(\chi_1^{l_1-1} \ldots \chi_r^{l_r-1}, i_1, \ldots, i_r| q_{h+1} + 1, \ldots, i_r| q_1, \ldots, q_h).$$

Indeed, the following map is surjective

$$H^0(\mathcal{A}(q_{1, \ldots, q_h}| (m_1 - i_1) + \ldots + (m_r - i_r) - r - 3)|H_{q_1} \cap \cdots \cap H_{q_h} \to$$

$$H^0(\mathcal{A}(q_{1, \ldots, q_h}| (m_1 - i_1) + \ldots + (m_r - i_r) - r - 3)|H_{q_1} \cap \cdots \cap H_{q_{h+1}} \to$$

(3.2.10)

because the cone in $H_{q_1} \cap \cdots \cap H_{q_h}$ over the hypersurface in $H^0(\mathcal{A}(q_{1, \ldots, q_h}| (m_1 - i_1) + \ldots + (m_r - i_r) - r - 3)|H_{q_1} \cap \cdots \cap H_{q_{h+1}})$ belongs to $H^0(\mathcal{A}(q_{1, \ldots, q_h}| (m_1 - i_1) + \ldots + (m_r - i_r) - r - 3)|H_{q_1} \cap \cdots \cap H_{q_{h+1}})$. Moreover for $q_i = i, i = 1, \ldots, r$, we have $F_X(i_1, \ldots, i_r| 1, \ldots, r)$ since to $(x, \chi) \in H^0(\mathcal{A}(q_{1, \ldots, q_h}|)$ corresponds the form $\psi = z_1^{m_1-i_1-1} \ldots z_r^{m_r-i_r-1} - \tilde{\phi}$ holomorphic on $\mathcal{F}_{m_1, \ldots, m_r}$ and satisfying $g^*(\psi) = \chi_1^{l_1-1} \ldots \chi_r^{l_r-1} \psi$. This shows (3.2.7) for $h = 1$ and that validity of (3.2.7) for the array $(q_1, \ldots, q_{h+1})$ provided it is valid for all $(q_1, \ldots, q_h)$.

**Step 3. An explicit formula for $F(i_1, \ldots, i_r)$.** Let $C(j_1, \ldots, j_s) = C_{j_1} \cup \cdots \cup C_{j_s}$ be a curve formed by a union of the components of $C$ and let

$$F_{C(j_1, \ldots, j_s)}(i_1, \ldots, i_s) = \dim H^1(\mathcal{A}(C(j_1, \ldots, j_s))_{i_1, \ldots, i_s}| (m_j - i_1) + \ldots + (m_s - i_s) + s - 3).$$

(3.2.11)

Note that

$$i_j = m_j - 1 (j \neq j_1, \ldots, j_s) \Rightarrow F(i_1, \ldots, i_r) = F_{C(j_1, \ldots, j_s)}(i_1, \ldots, i_s)$$

since the local conditions defining both sheaves coincide, indeed:

$$\frac{z_1^{m_1-i_1-1} \ldots z_r^{m_r-i_r-1} dx \wedge dy}{z_1^{m_1-1} \ldots z_r^{m_r-1}} = \frac{z_1^{m_1-i_1-1} \ldots z_r^{m_r-i_r-1} dx \wedge dy}{z_1^{m_1-1} \ldots z_r^{m_r-1}} ,$$

as well as the degrees of the curves in the corresponding linear systems. Moreover

$$F_{C(j_1, \ldots, j_s)}(0, \ldots, 0)$$
is the irregularity of the cover of \( \mathbb{P}^2 \) branched over \( C(j_1, \ldots, j_s) \) and having the ramification index \( m_j \) over the component \( C_i (i = j_1, \ldots, j_s) \).

We solve the recurrence relation (3.2.6) subject to the “initial condition” (3.2.11). It is convenient to view each relation (3.2.6) as the one connecting the values of the function defined at the vertices of the integer lattice in the parallelepiped \( 0 \leq x_i \leq m_i, (i = 1, \ldots, r) \). Each equation connects the values of this function at the vertices of a parallelepiped with sides equal to 1. It is clear that the sum of all equations (3.2.6) yields:

\[
F(0, \ldots, 0) = \sum_{0 \leq i \leq m_i - 1} s(i_1, \ldots, i_r) + \sum_{i_1 < \cdots < i_r} F_C(\ldots, 0)(0, \ldots, 0)
\]

\[
F(0, \ldots, 0) = \sum_{0 \leq i_1 \leq m_1 - 1} \varepsilon(\ldots, i_r) s(i_1, \ldots, i_r) + \sum_{i_1 < \cdots < i_r} F_C(\ldots, 0)(0, \ldots, 0)
\]

(3.2.12)

Remark. Alternative derivation of (3.2.12).

Sheaves \( \mathcal{A}_C (a_1 + \cdots + a_r - [\sum \frac{d_i(i+1)}{m_i}]) \) admit the following interpretation also yielding (3.2.12). Let us consider the global version of the diagram (2.3.1):

\[
\begin{array}{ccc}
\bar{\nu}_{m_1, \ldots, m_r} & \rightarrow & Y_C \\
\downarrow^\delta & & \downarrow^\rho \\
\nu_{m_1, \ldots, m_r} & \rightarrow & \mathbb{P}^2
\end{array}
\]

(3.2.13)

Here \( \rho : Y_C \rightarrow \mathbb{P}^2 \) is an embedded resolution of singularities of \( C \) which are worse than nodes, \( \bar{\nu}_{m_1, \ldots, m_r} \) is the normalization of \( \nu_{m_1, \ldots, m_r} \times_{\mathbb{P}^2} Y_C \) and \( \bar{\pi}, \bar{\rho} \) are the obvious projections. Let

\[
\tilde{\pi}_* (O_{\bar{\nu}_{m_1, \ldots, m_r}}) = \bigoplus \mathcal{L}^{-1}_{\chi^i \cdots \chi^r}
\]

(3.2.14)

be the decomposition by the characters of the Galois group acting on \( \tilde{\pi}_* (O_{\bar{\nu}_{m_1, \ldots, m_r}}) \).

Then we have:

\[
\mathcal{A}_C \left( \sum_{j} a_j - \left[ \sum_{j} \frac{d_j(i+1)}{m_j} \right] \right) = \rho_* (\tilde{\pi}_* (O_{\bar{\nu}_{m_1, \ldots, m_r}}) \otimes L_{\chi(m_{i-1} - (i+1), \ldots, m_r - (i+1)))}
\]

(3.2.15)

where \( \Delta \) is the polytope of quasiajunction containing \( (\frac{i+1}{m_1}, \ldots, \frac{i+1}{m_r}) \). Indeed it follows from (3.2.8) that a germ \( \phi \) of a holomorphic function belongs to the sheaf in the left side of (3.2.15) if and only if the order of \( \phi \) along an exceptional curve \( E_k \subset Y_C \) satisfies: \( \text{ord}_{E_k} \phi \geq \sum a_j \left( \frac{m_j}{m_j} - (i+1) \right) - c_k \) and the sheaf on the left is a subsheaf of \( O_{\mathbb{P}^2} (\sum_j d_j \left( \frac{m_j - (i+1)}{m_j} \right)) \) with the quotient having a zero-dimensional support. One readily sees that the sheaf on the right has the same local description. This identity also implies (3.2.12) as follows from Serre’s duality and (3.2.14).

Step 4. A vanishing result.

If \( \Delta \) is a polytope of quasiajunction, \( \Xi_k = \{(x_1, \ldots, x_r) \in U | k \leq \sum_{j=1}^r x_j \leq d_1 x_1 + \cdots + d_r x_r < k + 1 \} \) and \( k \) is such that \( \Delta \cap \Xi_k \neq \emptyset \) then

\[
H^1 (\mathcal{A}_C (d - r - 2 - k)) = 0
\]

(3.2.16)

unless \( \Delta \) is a contributing polytope of quasiajunction and \( \Delta \cap \Xi_k \) is a face of quasiajunction.

If \( \Delta \) isn’t contributing (cf. 2.6.2), then the intersection \( \Delta \cap \Xi_k \) has a positive volume. If \( X(n) \) is the number of points \( (\frac{i_1}{n}, \ldots, \frac{i_r}{n}) \) in \( \Xi_k \), then it follows from (3.2.12) that we have \( b_1 (C, n) \geq \dim H^1 (\mathcal{A}_C (d - r - 2 - k)) . X(n) \). We have \( X(n) > C \cdot n^r \) for some non zero constant \( C \). Therefore we get contradiction with Corollary (1.3.3) unless \( \dim H^1 (\mathcal{A}_C (d - r - 2 - k)) = 0 \).

Step 5. End of the proof. Step 4 and the formula (3.2.12) give a) and b) of the theorem in the case \( m_i \geq d_i \) for \( i = 1, \ldots, r \).

If \( \chi \) is a character of \( \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \) acting on \( H^0 (\Omega_{\nu_{m_1, \ldots, m_r}}) = H^1, 0 (\tilde{\nu}_{m_1, \ldots, m_r}) \) then \( \bar{\chi} \bar{\nu} \) is a character with eigenspace of the same dimension for the action of \( \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \) on \( H^0 (\bar{\nu}_{m_1, \ldots, m_r}) \). Hence part b) and Sakuma formula (cf. 1.3.2) imply that a points \( \left( \frac{i_1}{m_1}, \ldots, \frac{i_r}{m_r} \right) \) belongs to a contributing face of \( \Delta \) or its conjugate if and only if \( \exp (2 \pi \sqrt{-1} \cdot \frac{i_1}{m_1}, \ldots) \) belongs to \( j \)-th characteristic variety with \( j = \dim H^1 (\mathcal{A}_C (d_1 + \cdots + d_r - r - 3 - k(\Delta))) \). Since a characteristic variety is translated by a point of finite order subtorus (cf. 1.4.2) this implies c). Now the remaining cases of the formula a) follows from Sakuma’s result (1.3.2.2).

3.3. EXAMPLES

In 2.6.4 we did describe systems of equations for faces of quasiajunction in the case of arrangements of lines. To determine if a set of solutions of the system corresponding to a face \( \delta \) actually corresponds to a component of characteristic variety one should

a) calculate the superabundance (3.1.2) of the corresponding linear system and
b) decide the “amount of translation” i.e. to normalize the system of equations so that the g.c.d. of minors of the left hand sides of (2.6.1) will be equal to one.

In any event, if superabundance is not zero, then clearly the component of characteristic variety will be a connected component of the subgroup given by the equations: \( \exp (L_\rho) = 1 \) with \( P \) running through all vertices singled out by the face of quasiajunction.

Example 1. Let us calculate the irregularity of the abelian cover of \( \mathbb{P}^2 \) branched over the arrangement \( L : uv(u-v)v = 0 \) and corresponding to the homomorphism
$H_1(\mathbb{P}^2 - L) = \mathbb{Z}^3 \to (\mathbb{Z}/n\mathbb{Z})^3$. The only nontrivial ideal of quasidiagonal is the maximal ideal of the local ring with corresponding polytope of quasidiagonal: $x + y + z > 1$. Hence the irregularity of the abelian cover is $\text{Card} \{ (i, j) | 0 < i < n, 0 < j < n, \frac{i}{n} + \frac{j}{n} + \frac{1}{n} = 1 \} = \dim H^1(J(3 - 3 - 1))$ where $J = \text{Ker} \mathcal{O} \to \mathcal{O}_P$ where $P : u = v = 0$. $J$ has the following Koszul resolution:

\[
0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to J \to 0
\]

which yields $H^1(J(-1)) = H^2(\mathcal{O}(-3)) = \mathbb{C}$. Now the counting points on $x + y + z = 1$ yields $n^2 - 3n^2 + 2 \over 2$ as the irregularity of the abelian cover.

**Example 2.** Let us consider the arrangement formed by the sides of an equilateral triangle $(x_1, x_2, x_3)$ and its medians $(x_4, x_5, x_6)$ arranged so that the vertices are the intersection points of $(x_1, x_2, x_4), (x_2, x_3, x_5)$ and $x_3, x_1, x_6$ respectively (Cevas arrangement cf. (Barthel, Hinzbruch and Hofer)). It has 6 lines, 4 triple and 3 double points. The polytopes of quasidiagonality are the connected components of the partition of $\mathcal{U} = \{ (x_1, \ldots, x_6) | 0 \leq x_i \leq 1, i = 1, \ldots, 6 \}$ by the hyperplanes:

\[
x_1 + x_2 + x_3 + x_4 = 1, x_2 + x_3 + x_5 = 1, x_3 + x_1 + x_6 = 1, x_4 + x_5 + x_6 = 1
\]

The only face of a polytope of quasidiagonal which belongs to a hyperplane $H_k : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = k, k \in \mathbb{Z}$ is formed by set of solutions of the system of all 4 equations (3.3.1). This face belongs to $H_2$ and is the only contributing face. Hence the irregularity is equal to $N \cdot \dim H^1(J(6 - 3 - 2))$ where $N$ is the number of solutions (3.3.1) of the form $x_i = \frac{1}{2}$. To calculate $H^1(J(6 - 3 - 2))$ notice that 4 triple points form a complete intersection of two quadrics. This yields $H^1(J(1)) = H^2(\mathcal{O}(-3)) = \mathbb{C}$.

By (3.3.1), the only essential torus is a component of subgroup:

\[
t_1t_2t_4 = 1, \quad t_2t_3t_5 = 1, \quad t_1t_5t_6 = 1, \quad t_4t_5t_6 = 1
\]

This subgroup has two connected components:

\[
(u, v, u^{-1}v^{-1}, u^{-1}v^{-1}, u^{-1}v^{-1}), (-u, -v, -u^{-1}v^{-1}, -u^{-1}v^{-1}, -u, v), u, v \in \mathbb{C}^*.
\]

The second component is a translation of the first by $(1, 1, 1, -1, -1, -1)$, a point of order 2. Since (3.3.1) admits an integral solution image under the exponential map of the contributing face does contains trivial character and hence the subgroup in (3.3.3) is the essential torus.

There are also 4 nonessential tori corresponding to each of triple points:

\[
t_1t_2t_3 = 1, t_1t_5t_6 = 1, \quad t_3t_4t_5 = 1, t_4t_5t_6 = 1, \quad t_2t_3t_5 = 1, t_2t_3t_6 = 1, \quad t_1t_4t_5 = 1, t_1t_4t_6 = 1.
\]

Let us consider the abelian cover of $\mathbb{C}^2$ corresponding to the homomorphism $H_1(\mathbb{C}^2 - C) \to (\mathbb{Z}/n\mathbb{Z})^6/\mathbb{Z}/n\mathbb{Z}$ (embedding of the quotiented subgroup is diagonal). Then each of five tori contributes the same number into irregularity equal to $n^2 - 3n^2 + 2$, i.e. the irregularity of the abelian cover of $\mathbb{P}^2(\mathbb{C})$ is $5n^2 - 3n^2 + 2$ (e.g., for $n = 5$ the irregularity is 30, cf. (Ishida)).

**Example 3.** Let us calculate the characteristic varieties of the union of 9 lines which are dual to nine inflection points on a non-singular cubic curve $C \subset \mathbb{P}^2(\mathbb{C})$. This arrangement in $\mathbb{P}^2(\mathbb{F})$ has 12 triple points corresponding to 12 lines determined by the pairs of the inflection points of $C$. One can view inflection points of $C$ as the points of $\mathbb{P}^2(\mathbb{F})$ (if $\mathbb{F}$ is the field with 3 elements) i.e. as the points of the affine part in a projective plane $\mathbb{P}^2(\mathbb{F})$. The triple points of this arrangement then can be viewed as lines in $\mathbb{P}^2(\mathbb{F})$ different from the line at infinity (i.e. the complement to the chosen affine plane). In dual picture one identifies triple points of this arrangement with points of the dual plane $\mathbb{P}^2(\mathbb{F})$ different from a fixed point $P$ corresponding to the line at infinity. The lines of this arrangement in $\mathbb{P}^2(\mathbb{C})$ are identified with the lines in $\mathbb{P}^2(\mathbb{F})$ not passing through the fixed point $P$.

Each essential component corresponds to a collection of vertices $S$ (cf. (2.6.4), example 2). The structure of the system of equations (2.6.1) shows that $|S|/k = r/m = 3$. Hence one has either:

a) $|S| = 3, k = 1 \text{ or } b) |S| = 6, k = 2 \text{ or } c) |S| = 9, k = 3 \text{ or } d) |S| = 12, k = 4$.

Cases a) and b) will not define non empty tori since in this case $r^2 > 9|S|$ (cf. corollary 4.1).

In the case c) each collection $S$ is determined by one of 4 choices of a line $\ell$ through $P$ and consists of 9 points in $\mathbb{P}^2(\mathbb{F})$ in the complement to the chosen line. In this case the corresponding homogeneous system has rank 7 i.e. a 2-dimensional space of solutions. Moreover, $\dim H^1(\mathbb{P}^2(\mathbb{C}), I(9 - 3 - 3)) = 1$ since the points on $\mathbb{P}^2(\mathbb{C})$ corresponding to 9 points in $\mathbb{P}^2(\mathbb{F})$ in the complement to a line $p \subset \mathbb{P}^2(\mathbb{F})$ form a complete intersection of two cubics. These cubics formed by the unions of triples of lines in $\mathbb{P}^2(\mathbb{C})$ corresponding to triple of lines in $\mathbb{P}^2(\mathbb{F})$ passing through a point of $\ell$. Indeed, for a given $P_1, P_2 \in \ell$ and a point $Q$ on $\mathbb{P}^2(\mathbb{F})$ outside of $\ell$, there are exactly 2 lines in $\mathbb{P}^2(\mathbb{F})$ intersecting at this point and passing respectively through $P_1$ and $P_2$. The same incidence relation is valid on $\mathbb{P}^2(\mathbb{C})$.

In the case d) the homogeneous system has rank 9, i.e. the corresponding system does not define a torus.

Non essential tori correspond to subarrangements with number of lines divisible by $m = 3$ (cf. 2.6.3). There are 12 triples of lines corresponding to each of triple points each defining a 2-torus. A collection of 6 lines should have 4 triple points but the arrangement of this example does not contain such subarrangements.

Therefore we have 16 2-dimensional tori. In the abelian cover of $\mathbb{C}^2$ which sends each generator of $H_1(\mathbb{C}^2 - C)$ to a generator of $\mathbb{Z}/n\mathbb{Z}$ contributing tori are the essential torus of this arrangement and subtori corresponding to subarrange-
ments formed by triple of lines defined by the triple points. Each torus contributes $(n - 1)(n - 2)/2$ to the Betti number, i.e. the total Betti number of this cover is $16 \times (n - 1)(n - 2)/2$. These tori can be explicitly described as follows. Defining equations of non essential tori are products of 3 generators $t_i$'s corresponding to a triple of points in $\mathbb{P}^2_3$ belonging to a line with the rest of $t_i$ is 1. Each of essential tori is given by 9 equations $t_if_k = 1$ where $(i, j, k)$ are the triples of points $\mathbb{P}^2_3$ (which interpreted as the lines of the arrangement) which belong to lines not passing through a fixed point at infinity.

If the point at infinity is $(1, -1, 0)$, then the lines not passing through it are: $x + z = 0, x - z = 0, x = 0, y = 0, x - y = z, 0, x - y - z = 0, y - z = 0$ i.e. the corresponding torus satisfies:

\[
t_{20}t_{21}t_{22} = 1, t_{00}t_{10}t_{11} = 1, t_{00}t_{01}t_{02} = 1, t_{00}t_{10}t_{20} = 1, t_{00}t_{11}t_{22} = 1, t_{01}t_{12}t_{20} = 1, t_{02}t_{12}t_{21} = 1, t_{10}t_{11}t_{21} = 1, t_{10}t_{12}t_{21} = 1, (3.3.5)
\]

where the points of the complement to $z = 0$ (i.e. the lines in $\mathbb{P}^2_3(\mathbb{C})$) are labeled as:

\[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2).\]

The corresponding torus can be parameterized as:

\[
t_{00} = t_{01} = s_{t2} = t - s, t_{10} = s, t_{11} = t - s, t_{12} = s.
\]

The equations for other essential tori, corresponding to choices of the point at infinity as respectively: $(1, 1, 0), (1, 0, 0), (0, 1, 0)$ can be obtained from (3.3.5) by applying linear transformation to the indices which takes $(1, -1, 0)$ to respective point. For $(x, y) \rightarrow (x, y)$ which takes $(1, -1, 0)$ to $(1, 0, 0)$ we obtain:

\[
t_{20}t_{22}t_{21} = t_{10}t_{12}t_{11} = t_{00}t_{02}t_{01} = t_{00}t_{01}t_{20} = t_{00}t_{12}t_{21} = t_{02}t_{12}t_{20} = t_{00}t_{10}t_{22} = t_{01}t_{11}t_{21} = 1.
\]

For $(x, y) \rightarrow (x, x+y)$ which takes $(1, -1, 0)$ to $(0, 0, 0)$ we obtain:

\[
t_{22}t_{20}t_{21} = t_{11}t_{12}t_{10} = t_{00}t_{01}t_{02} = t_{00}t_{11}t_{22} = t_{00}t_{12}t_{21} = t_{01}t_{01}t_{20} = t_{02}t_{12}t_{20} = t_{02}t_{12}t_{20} = t_{01}t_{12}t_{21} = 1.
\]

For $(x, y) \rightarrow (y, y, y)$ which takes $(1, -1, 0)$ to $(0, 0, 1)$ we have:

\[
t_{20}t_{01}t_{12} = t_{10}t_{21}t_{01} = t_{00}t_{01}t_{12} = t_{00}t_{10}t_{20} = t_{00}t_{21}t_{12} = t_{11}t_{01}t_{20} = t_{22}t_{01}t_{01} = t_{22}t_{02}t_{12} = t_{11}t_{21}t_{01}.
\]

Example 4. Let us consider the curve of degree 4 which has one ordinary point of multiplicity 4. Faces of the polytopes of quasiadjunction are $H_1 : x_1 + x_2 + x_3 + x_4 = 1$ (resp. $H_2 : x_1 + x_2 + x_3 + x_4 = 2$). The number of points $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n})$ on $H_1$ (resp. $H_2$) is $(n - 1)(n - 2)(n - 3)/6$ (resp. $(n - 1)(2n^2 - 4n + 3)/3$). The ideal corresponding to the polytope of quasiadjunction with the face $H_1$ (resp. $H_2$) is $\mathcal{M}^2$ (resp. $\mathcal{M}$ the maximal ideal of the local ring) and the level of the supporting face $H_1$ (resp. $H_2$) is 1 (resp. 2). Moreover dim $H^1(\mathbb{P}^2_3, \mathcal{M}^{-d}(4 - 3l - 1))$ is 2 (resp. 1) for $l = 1$ (resp. $l = 2$). Hence the irregularity of the cover corresponding to homomorphism $H_1(\mathbb{P}^2_3 - U_{i=1,2,3,4}Li) \rightarrow \mathbb{Z}/n\mathbb{Z}$ is equal to

\[
\frac{(n - 1)(n - 2)(n - 3)}{6} + \frac{(n - 1)(2n^2 - 4n + 3)}{3} + \frac{(n - 1)(n - 2)}{2} = (3.3.6)
\]

This implies that the characteristic variety in this case is just $t_{12}t_{20}t_{21} = 1$.

The latter contains $(n - 1)^3 - (n - 1)(n - 2) = (n - 1)(n^2 - 3n + 3)$ points with coordinates in $\mu_n$ and the Betti number of the branched cover from Sakuma's formula is equal to $2(n - 1)(n^2 - 3n + 3) + 4(n - 1)(n - 2) = 2(n - 1)(n^2 - n - 1)$.

Example 5. Let us consider the arrangement formed by 12 lines which compose 4 degenerate fibers in a Hesse pencil of cubics formed by a non singular cubic curve and its Hessian. For example one can take the following pencil:

\[
x^3 + y^3 + z^3 - 3\lambda xyz = 0.
\]

This arrangement has 9 points of multiplicity 4 (infection points of non singular cubic). In $\mathbb{C}^{12}$ there are 10 tori of dimension 3 which are defined by 9 quadruples of lines corresponding to 9 quadruple points and one 3-torus corresponding to the whole configuration. Contribution into the first Betti number an abelian cover also comes from 94 tori of dimension 2: 2-tori corresponding to triples of lines forming each of 9 quadruple points (total 36 2-tori), 2-tori corresponding to configurations of 9 lines formed by triples of 4 special fibers of the pencil (total 4 2-tori) and 54 2-tori corresponding to configurations of 6 lines passing through 4 inflection points no three of which belong to a line (since the choice of 4 points must be made among points of affine space over $\mathbb{F}_3$ the ordered collection can be made in $9 \times 8 \times 6 \times 3$ way and $54 = 9 \cdot 8 \cdot 6 \cdot 3/24$). In particular, the irregularity of the cover with the Galois group ($\mathbb{Z}/2\mathbb{Z})^2$ is equal to 154, cf. (Ishida). Indeed the contribution of each 2-torus into the first Betti number is 2 and in the case of 3-tori the contribution is 6, since the 3-torus contains 6 points with coordinates $i/3$. Since the depth of 3-torus is 2 the first Betti number is equal to $6 \times 10 \times 2 + 94 \times 2$.

4. The structure of characteristic varieties of algebraic curves

In this section we describe sufficient conditions for the vanishing of cohomology of linear systems which appear in description of characteristic varieties given in
section 3. This, therefore, yields conditions for absence of essential components. In the cyclic case one obtains triviality of Alexander polynomial.

4.1. ABSENCE OF CHARACTERISTIC VARIETIES FOR CURVES WITH SMALL NUMBER OF SINGULARITIES

**THEOREM 4.1.1.** Let \( C \) be a plane curve as above. Suppose that \( \rho : Y \to \mathbb{P}^2 \) is obtained by a sequence of blow ups such that the proper preimage \( \tilde{C} \) of \( C \) in \( Y \) has only normal intersection with the exceptional set and satisfies \( \tilde{C}^2 > 0 \). Then \( C \) has no essential characteristic subvarieties.

**COROLLARY 4.1.2.** 1. Let \( C \) be an irreducible curve which has ordinary cusps and nodes as the only singularities. If the number of cusps is less than \( d^2/6 \) then the Alexander polynomial of \( C \) is equal to 1.

2. Let \( \mathcal{H} \) be an arrangement consisting of \( d \) lines and which has \( N \) points of multiplicity \( m \). Let \( l(\delta) \) be the level of a face of quasiadjunction for the complement to \( \mathcal{H} \). If \( d^2 > m^2 N \) then the superabundance is zero for the system of curves of degree \( d - 3 - l(\delta) \) which local equations belong to the ideal of quasiadjunction corresponding to \( \delta \) at the points of multiplicity \( m \).

**Remark.** One can compare corollary 1 with results of (Nori). The latter yields that the fundamental group of the complement to a curve of degree \( d \) with \( \delta \) nodes and \( \kappa \) cusps is abelian if \( d^2 > 6\kappa + 2\delta \) while a weaker inequality \( d^2 > 6\kappa \) yields the triviality of the Alexander polynomial. For example, for the branching curve of a generic projection of a smooth surface of degree \( N \) in \( \mathbb{P}^3 \) one has \( d^2 > 6\kappa \) for \( N > 4 \) but \( d^2 < 6\kappa + 2\delta \) for \( N > 2 \). The fundamental groups of these curves are non abelian for \( N > 2 \) and the Alexander polynomial for \( N = 3, 4 \) is equal to \( t^2 - t + 1 \), cf. (Libgober, 1983).

**PROOF OF THE THEOREM.** We should show that for any contributing face of the quasi-adjunction \( \delta \) we have \( \dim H^1(\mathbb{P}^2, \mathbb{A}_5(\sum d_i - 3 - l(\delta))) = 0 \). If \( \rho : Y \to \mathbb{P}^2 \) is a blow up of \( \mathbb{P}^2 \), satisfying conditions of the theorem, then we have:

\[
\mathbb{A}_5(\sum d_i - 3 - l(\delta)) = \rho^*(\omega_Y \otimes \mathcal{O}_Y(\gamma \tilde{C}) \otimes \mathcal{O}_Y(\sum \varepsilon_k E_k))
\]

(4.1.1)

for some rational \( \gamma > 0 \) and \( 0 \leq \varepsilon < 1 \). More precisely, \( \gamma = \frac{1}{\varepsilon_k \rho_1 \cdot \sum d_i(1 - \frac{j_i + 1}{m_i})} \) for some choice of \((\ldots, (j_i + 1)/m_i, \ldots)\) belonging to the face \( \delta \) (with \( \varepsilon_k \) a priori depending on this choice). Indeed, from (2.3.6) and the discussion after, the multiplicity \( f_k(\phi) \) along an exceptional curve \( E_k \) of the pull back on \( Y \) of a germ in the ideal of quasiadjunction with parameters \((j_1, \ldots, j_r, m_1, \ldots, m_r)\) such that \((j_1 + 1)/m_1, \ldots, (j_r + 1)/m_r) \in \delta \) satisfies:

\[
f_k(\phi) \geq \left[ \sum a_{i,j} \left( 1 - \frac{j_i + 1}{m_i} \right) - c_k \right].
\]

(4.1.1)

Hence

\[
A_5 = \rho^*(\otimes \mathcal{O}_Y \left( \left( c_k - \left[ \sum a_{i,j} - a_{i,1} \left( \frac{j_i + 1}{m_i} \right) \right] E_k \right) \right)).
\]

(4.1.1.2)

We have: \( l(\delta) = \sum d_i - 3 - l(\delta) \) and \( \mathcal{O}_{\mathbb{P}^2}(C_i) = \mathcal{O}_{\mathbb{P}^2}(d_i) \). Therefore

\[
A_5 \left( \sum d_i - 3 - l(\delta) \right) = \mathcal{O}_{\mathbb{P}^2} \left( \sum C_i \left( 1 - \frac{j_i + 1}{m_i} \right) \right) \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \otimes \rho^*(\otimes \mathcal{O}_Y \left( \left( c_k - \left[ \sum a_{i,j} - a_{i,1} \left( \frac{j_i + 1}{m_i} \right) \right] E_k \right) \right)).
\]

(4.1.1.3)

Since \( \omega_Y = \otimes \mathcal{O}_Y (c_k E_k) \otimes \rho^*(\mathcal{O}_{\mathbb{P}^2}(-3)) \) and

\[
\otimes \mathcal{O}_Y (a_{i,j} \left( 1 - \frac{j_i + 1}{m_i} \right) E_k) \otimes \mathcal{O}_{\tilde{C}} \left( \frac{d_i}{m_i} \right) = \rho^*(\mathcal{O}_{\mathbb{P}^2}(C_i \left( 1 - \frac{j_i + 1}{m_i} \right)))
\]

(4.1.1.4)

(because \( \mathcal{O}_{\mathbb{P}^2}(C_i) = \mathcal{O}_{\mathbb{P}^2}(d_i) = \mathcal{O}_{\mathbb{P}^2}(C_i \Sigma_{\mathbb{P}^2}) \)), we see that (4.1.1.3) yields (4.1.1) with \( \varepsilon_k = \left( \sum a_{i,j} - a_{i,1} \left( \frac{j_i + 1}{m_i} \right) \right) \) where \( \{x\} = x - \lfloor x \rfloor \) is the fractional part.

The Kawamata-Viehweg vanishing theorem (cf. for example (Kollar)) implies that the cohomology of the sheaf \( \omega_Y \otimes \mathcal{O}_Y (\gamma \tilde{C}) \otimes \mathcal{O}_Y (\Sigma \varepsilon_k E_k) \) is trivial in positive dimensions if \( \tilde{C} \) is big and nef. But this follows from the assumptions of the theorem. Finally, the exact sequence \( 0 \to E_2^\bullet \to H^1(\mathcal{Y}, \mathcal{F}) \to E_2^{r+1}(\mathcal{Y}, \mathcal{F}) \) of lower degree terms in the Leray spectral sequence \( H^1(\mathbb{P}^2, R^1 \rho_* \mathcal{F}) \to H^0(\mathcal{Y}, \mathcal{F}) \) for the sheaf \( \mathcal{F} = \omega_Y \otimes \mathcal{O}_Y (\gamma \tilde{C}) \otimes \mathcal{O}_Y (\Sigma \varepsilon_k E_k) \) yields \( H^1(\mathbb{P}^2, A_5(\sum d_i - 3 - l(\delta))) = 0 \).

**PROOF OF THE COROLLARY.** For each blow up at an ordinary point of multiplicity \( m \) of a curve \( C \) we have \( \tilde{C}^2 = C^2 - m^2 \) where \( \tilde{C} \) is the proper preimage of \( C \). Hence \( \mathcal{O}(\mathcal{H}) \) is big if \( d^2 > m^2 N \). The case of ordinary cusps is similar.

4.1.2. Dimensions of components of characteristic varieties

This theorem imposes restrictions on the dimensions of the contributing faces and hence on the dimensions of characteristic varieties. For example, let us consider an arrangement of lines with at most triple points as singularities. Then each contributing face is the intersection of hyperplanes defining the only local polytope of an ordinary triple point. These hyperplanes are given by the equations of the form \( x_i + x_j + x_k = 1 \) where \((i, j, k)\) are the indices corresponding to the lines through the triple point. The matrix of this system therefore has the property that in each row only 3 non zero entries are equal to 1, any two rows have at most one non zero entry in the same column and the number of rows is at least \( d^2/9 \) (since by the corollary only in this case one can get a contributing face with \( H^1 \neq 0 \)). In particular, the number of non zero entries in the matrix is at least \( d^2/3 \). The rank of this system is at least \( d/3 \). Indeed, the matrix contains a column with at least \( d/3 \).
PROPOSITION 4.2.2. a) Let $\delta$ and $\delta'$ be two faces of global polytopes quasiadjunction such that the Zariski closures of $\exp(\delta)$ and $\exp(\delta')$ coincide. Then if $\delta$ is a contributing face then $\delta'$ is also contributing and $H^1(A_0(d - 3 - l(\delta))) = H^1(A_0(d - 3 - l(\delta'))).

b) Let $\alpha \in \mathbb{Q}$ be such that $\alpha \cdot \text{gcd}(d_1, \ldots, d_r)$ is the index of a face of quasiadjunction $\delta$ and $\sigma \in \text{Gal}(\mathbb{Q}(\exp(2\pi i \alpha)) / \mathbb{Q})$ such that $\sigma(\exp(2\pi i \alpha)) = \exp(2\pi i \beta)$ with $0 < \beta < 1$. Then $\beta$ is equal to $\frac{\sigma \cdot \text{gcd}(d_1, \ldots, d_r)}{\text{gcd}(d_1, \ldots, d_r)}$ for some face of quasiadjunction $\delta'$ and $H^1(A_0(d - 3 - l(\delta'))) = H^1(A_0(d - 3 - l(\delta'))).

PROOF. a) Since Zariski closures of $\exp(\delta)$ and $\exp(\delta')$ are the same the corresponding to $\delta$ and $\delta'$ components of the characteristic variety are the same. If the depth of this component of characteristic variety is $i$, then the dimension of each of the cohomology group in the statement equals $i$ and the result follows.

b) Since $i$-th characteristic variety is defined over $\mathbb{Z}$, the Galois group $\text{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts on the set of its irreducible components. The irreducible component corresponding to $\delta$ is a translation by $\exp(2\pi \sqrt{-1} \alpha)$ of a subgroup of $H^1(C^2 - C, C^r)$ defined over $\mathbb{Z}$, therefore $\sigma$ takes the component corresponding to $\delta$ into translation of the same subgroup by $\exp(2\pi \sqrt{-1} \beta)$. This translation is a Zariski closure of $\exp(2\pi \sqrt{-1} \delta')$ for some face $\delta'$. It does satisfy the conclusions of b.)

Remarks.
1. For irreducible curves the order of each face of quasiadjunction is a root of a local Alexander polynomial. So the divisibility theorem from (Libgober, 1982) is a special case of 4.2.1.
2. One of the consequences of 4.2.1 is a non-trivial restriction on an abstract group which is necessary to satisfy in order that the group can be realized as the fundamental group of an arrangement. For example $t_1 \cdots t_r = -1$ cannot be a component of a characteristic variety of arrangement of $r$ lines since it is cannot belong to an intersection of subgroups of $C^r$.
3. As an illustration to 4.2.2b), let us consider an irreducible curve of degree $d$ with singularities locally isomorphic to singularity $x^2 = y^2$. If the linear system consisting of curves ofdegree $d - 3 - d/10$ with local equations belonging to ideals of quasiadjunction of all singular points corresponding to the constant of quasiadjunction $1/10$ is superabundant, then the linear system of curves of degree $d - 3 - 3d/10$ with local equations in the ideals of quasiadjunction corresponding to $3/10$ is also superabundant and the superabundances are equal.
4. An example of faces of quasiadjunction with the same Zariski closure of the images of the exponential map as in a) of the proposition is given by $x_1 + \cdots + x_m = i, x_1 + \cdots + x_m = j, 0 < i, j \leq m - 2$ which are the faces of quasiadjunction for the complement to $m$ lines through a point (cf. 2.5 example 3.).
5. Resonance conditions for rank one local systems on complements to line arrangements

5.1. COMPLEXES ASSOCIATED WITH ARRANGEMENTS

Let \( L = \bigcup_{i=1}^{n} L_i \) be an arrangement of lines in \( \mathbb{C}^2 \). We shall assume for convenience (cf. (1.2.3)) that the line at infinity is transversal to all lines in \( L \). Let \( l_i(x,y) = 0 \) be the equation of \( L_i \) and \( \eta_i = \frac{1}{2\pi i} \frac{dl_i}{l_i} \). Let \( A^i(L) = \langle \eta_i, \eta_{i+1}, \ldots, \eta_{j-1}, \eta_j \rangle \) be the subspace generated by the forms \( \eta_i \wedge \cdots \wedge \eta_j \) in the space \( H^0(\Omega^i(L)) \) of meromorphic forms with poles along \( L \). Let

\[
\omega = \sum \eta_i \cdot s_i, s_i \in \mathbb{C}
\]

(5.1.1)

The exterior product with \( \omega \) defines the complex:

\[
A^*_\omega : 0 \to A^0 \to A^1 \to A^2 \to 0
\]

(5.1.2)

If \( s_i = 0, (i = 1, \ldots, r) \), then the cohomology groups of \( A^* \) are isomorphic to the cohomology groups of \( \mathbb{C}^2 - L \) (Brieskorn). On the other hand, the collection \( s_* = (s_1, \ldots, s_r) \) defines the map \( \pi_1(\mathbb{C}^2 - L) \to \mathbb{C}^r \) which sends \( \gamma_i \) (cf. (1.1)) to \( \exp(2\pi i \gamma_i) \) and hence the local system which we shall denote \( A_\omega \). A theorem from (Esnault, Schechtem and Viehweg, p. 558), in the case of line arrangements, asserts that

\[
H^i(A_\omega) = H^i(A^*_\omega)
\]

(5.1.3)

provided the following non resonance condition is satisfied. For any point singular point \( P \) of \( L \) of multiplicity \( m > 2 \), if the lines through \( P \) are \( l_{i_1}, \ldots, l_{i_m} \), then

\[
s_i_1 + \cdots + s_i_m \neq n \in \mathbb{N} - 0
\]

(5.1.4)

Theorem 5.1.1. The isomorphism (5.1.3) takes place, provided

\[
(\exp(2\pi i s_1), \ldots, \exp(2\pi i s_r))
\]

does not belong to the characteristic variety \( \text{Char}_1 \) of \( \mathbb{C}^2 - L \).

5.2. REMARKS

1. It is easy to construct examples of local systems for which (5.1.4) is violated but for which (1.5.3) takes place. Indeed, the image under the exponential map onto the torus \( \mathbb{C}^r \) of those \( (s_1, \ldots, s_r) \) which violate (5.1.4) is a union of codimension 1 tori in \( \mathbb{C}^r \). On the other hand, the characteristic varieties typically have rather small dimension relative to \( r \) (cf. (3.3) and (4.1)).

2. For arrangements of arbitrary dimension in \( \mathbb{C}^n \) (with \( l_i \) denoting the equations of hyperplanes of the arrangement, rather than lines) we have

\[
H^1(A^*(L), \omega) = 0
\]

(5.2.1)

PROOF. If \( \dim H^1(A^*(L), \omega) > 0 \), i.e. there exist linearly independent with \( \omega \) form \( \eta \) such that \( \eta \wedge \omega = 0 \), then for any form \( \omega' \) in the space spanned by \( \omega \) and \( \eta \) one has \( \dim H^1(A^*(L), \omega') > 0 \). Let \( \mathcal{V} \) be an irreducible component of \( \mathcal{V}_1 \) containing \( \omega \) and having dimension \( k \geq 2 \). For the local system \( L_\omega \) corresponding to each \( \omega' \in \mathcal{V} \) we have \( \dim H^1(L_\omega) > 0 \). Indeed we can assume that \( \omega' \) is generic since this only decrease \( H^1(L_\omega) \), cf. (Libgober and Yuzvinsky).
other hand for generic \( \omega \), according to (Essnault, Schechtman and Viehweg), we have 
\( \dim H^2(A^*, \omega^\prime) = \dim H^1(L_{\omega}) \). Therefore \( L_{\omega} \) belongs to an irreducible component, say \( V \), of the characteristic variety of \( C^2 - L \). Since the exponential map is a local homomorphism this component has the dimension equal to at least \( k \). In fact the dimension of this component is exactly \( k \). Assume to the contrary that this dimension is \( l > k \) and let \( f : C^2 - L \rightarrow \mathbb{P}^1 \cup \bigcup_{i=1}^{\text{dim} + k+1} p_i \) be the map on a curve of general type (cf. (1.4.2) and (Arapura, Prop. 1.7)) corresponding to the component \( V \). Then the pull back of form \( H^0(\mathbb{P}^1, \Omega^1(\log(\cup p_i))) \) gives \( l \)-dimensional space of forms on \( C^2 - L \) for which the wedge with \( \omega \) is zero (note that the map \( \pi^* \) is injective on \( H^1 \)) and we have a contradiction. Let \( i_1^{(j)} \cdots i_{\text{dim} + k}^{(j)} = 1 (j = 1, \ldots, s) \) be the equations defining \( V \) (cf. 4.2). Then \( \omega \) belongs to the union of affine subspaces of \( H^0(\mathbb{P}^2, \Omega^1(\log(\bigcup_{i=1}^{\text{dim} + k} p_i'))) \) given by \( q_i = \sum a_{i,j} x_j = n_j, n_j \in \mathbb{Z}, j = 1, \ldots, s \). Since 
\( \dim H^1(A^*, \lambda_\omega) = \dim H^1(A^*, \omega), \lambda \in C^* \) we see that \( n_j = 0 \) for any \( j \). Hence \( \psi \) is a linear space of dimension \( k \) (and \( i = \dim H^1(A^*, \omega) = \dim H^1(L_{\omega}, \omega) = k - 1 \)).

5.5. COMBINATORIAL CALCULATION OF CHARACTERISTIC VARIETIES

A consequence of the Theorem 5.4 is that the irreducible components of the characteristic varieties containing the identity element of \( H^1(C^2 - L, C^\ast) \) are determined by the cohomology of the complex (5.1.2), \( H^1(A^*, \omega) \) is the quotient of \( \{ \eta \in A^1 | \eta \wedge \omega = 0 \} \) by the subspace spanned by \( \omega \) and can be calculated as follows.

It is easy to see that a 2-form is cohomologous to zero iff its integrals over all 2-cycles belonging to small balls about the multiple points of the arrangement are zeros. The group of such 2-cycles near a point which is the intersection of the lines \( l_1, \ldots, l_n \) is generated by \( y_x \times (y_{l_1} + \cdots + y_{l_n}) \). If \( \sum A_{\eta_j} \wedge \omega \) is cohomologous to zero in \( \Omega^2(C^2 - L) \) then vanishing of \( \int A_{\eta_j} \wedge \eta \wedge s_i \) over these 2-cycles yields:

\[
A_j(\sum s_i) - (\sum A_j)s_j = 0 \quad (5.5.1)
\]

Therefore we obtain

\[
A_j = C_\omega s_j, \quad \text{if} \quad \sum s_j \neq 0, \quad (5.5.2)
\]

\[
\sum A_j = 0, \quad \text{if} \quad \sum s_j = 0. \quad (5.5.3)
\]

for vertices \( v \) of the arrangement. If we are looking for essential components of the characteristic variety (which we always can assume) then \( s_i \neq 0 \) and condition (5.5.2) can be replaced by

\[
\frac{A_j}{s_j} = \frac{A_l}{s_l}, \quad \text{if} \quad \sum s_j \neq 0. \quad (5.5.4)
\]

Now for each subset of the set of vertices such that the system of equation (5.5.3) and supplementing it by equations (5.5.4) for vertices outside of selected subset has a solution non proportional to \( (s_1, \ldots, s_n) \) we obtain a component \( V \) and hence corresponding component of the characteristic variety. We leave as an exercise to the reader to work out calculations of the characteristic varieties for the examples from section 3 using this method.

References


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COMMUNICATION NETWORKS AND HILBERT MODULAR FORMS

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Abstract. Ramanujan graphs, defined and constructed by Lubotzky, Phillips and Sarnak, allow the design of efficient communication networks. In joint work with B. Jordan we gave a higher-dimensional generalization. Here we explain how one could use this generalization to construct efficient communication networks which allow for a number of verification protocols and for the distribution of information along several channels. The efficiency of our network hinges on the Ramanujan-Petersson conjecture for certain Hilbert modular forms. We obtain this conjecture in sufficient generality to apply to some particularly appealing constructions, which were not accessible before.

Key words: Ramanujan local systems, cubical complexes, quaternion algebras, spectrum of the Laplacian.

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Introduction

The concept of a Ramanujan graph was introduced and studied by (Lubotzky, Phillips and Sarnak), shortly (LPS). These are $r$-regular graphs for which the nontrivial eigenvalues $\lambda \neq \pm k$ of the adjacency matrix satisfy the bounds $|\lambda| \leq 2\sqrt{r-1}$. In many aspects these bounds are optimal and natural. For example, the adjacency matrix is the combinatorial analog of the Laplace operator, and the bounds parallel the (conjectured) Selberg bounds for the Laplacian on Riemann surfaces. The main result of (LPS) was an explicit construction of $p+1$-regular such graphs, $p \equiv 1 \pmod 4$ a prime, through the arithmetic of quaternion algebras over the rational numbers. From the point of view of Communication Network Theory, the arithmetic examples are particularly interesting: all Ramanujan graphs are super-expanders; but in addition the examples have many other useful properties, for example very good expansion constants and large girth. Thus they can be used to design efficient communication networks.

The Ramanujan property for the (LPS) examples hinges on the truth of the Ramanujan-Petersson conjecture for an appropriate space of modular forms of weight 2 over $\mathbb{Q}$. Let $f$ be a weight 2 holomorphic cuspidal Hecke eigenform on