Math 215: Introduction to Advanced Mathematics
Problem Set 11

Due: Monday December 1

1) a) Prove that the interval $(0, 1)$ is equipotent with the interval $(a, b)$.
   [Note: the interval $(c, d) = \{x \in \mathbb{R} : c < x < d\}$.]
   b) Prove that the interval $(0, 1)$ is equipotent with the interval $(0, +\infty)$.
   c) Prove that $\mathbb{R}$ is equipotent with the interval $(0, +\infty)$. Conclude that $(0, 1)$ is equipotent with $\mathbb{R}$.
   [HINT: For this problem you can use familiar functions from algebra and calculus.]

2) If $A_1, A_2, \ldots$ are sets we let

   \[ A = \bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for some } i = 1, 2, \ldots\}. \]

   Suppose each $A_i$ is countable and $f_i : \mathbb{N} \to A_i$ is a surjection. Let $f : \mathbb{N} \times \mathbb{N} \to A$ be the function $f(i, j) = f_i(j)$.
   a) Prove that $f$ is a surjection.
   b) Prove that $A$ is countable. We have proved that a countable union of countable sets is countable.

3) If $A$ is any set we define $A^2 = A \times A$ and

   \[ A^n = \underbrace{A \times \ldots \times A}_{n \text{-times}}. \]

   We also think of $A^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A\}$. Let $Seq(A) = \bigcup_{n=1}^{\infty} A^n$.

   Then $Seq(A)$ is the set of all finite sequences from $A$.
   a) Prove that if $A$ is countable, then $A^n$ is countable for all $n$. [Hint: This should be an easy induction.]
   b) Prove that if $A$ is countable, then $Seq(A)$ is countable. [Hint: Use 2)]

(5pt bonus) If $f : A \times B \to C$. For each $a \in A$, we get a function $f_a \in \mathcal{F}(B, C)$, where $f_a : B \to C$ is given by $f_a(b) = f(a, b)$. 

Let $\Phi : \mathcal{F}(A \times B, C) \to \mathcal{F}(A, \mathcal{F}(B, C))$ be defined so that for $f : A \times B \to C$, $\Phi(f) : A \to \mathcal{F}(B, C)$ is the function such that $\Phi(f)(a) = f_a$.

Prove that $\Phi$ is a bijection. This gives an argument that even for infinite sets

$$
\left(|C|^{|A|}\right)^{|B|} = |C|^{|A||B|}.
$$