Comparing Nash and IESDS equilibria

We consider a finite two-player game.

**Proposition 1** If \((a^*, b^*)\) is a strictly dominant strategy equilibrium, then \((a^*, b^*)\) is a Nash equilibrium.

**Proof** The strategy \(a^*\) strictly dominates each strategy in \(A_1\). Thus

\[
v_1(a^*, b^*) > v(a, b^*)
\]

for all \(a \in A_1\) with \(a \neq a^*\). Thus \(a^*\) is the unique best response to \(b^*\), i.e.,

\[
BR_1(b) = \{a^*\}.
\]

Similarly, \(b^*\) is the unique best response to \(a^*\). Thus \((a^*, b^*)\) is a Nash equilibrium. \(\square\)

The game *Battle of the Sexes* shows that the converse fails as there are games with Nash equilibria which are not strictly dominant.

**Proposition 2** If \((a^*, b^*)\) is a Nash equilibrium, then \((a^*, b^*)\) is an IESDS-equilibrium.

**Proof** We prove this by contradiction. Suppose \((a^*, b^*)\) is not an IESDS-equilibrium. Then one of the strategies is removed at some stage of the construction. Let’s suppose that \(a^*\) is removed no later than \(b^*\) (the other case is similar). Consider the stage when \(a^*\) is eliminated. At this stage of the construction, we have a game where \(a^*\) and \(b^*\) are possible strategies and, because it is about to be eliminated, there is a strategy \(a' \in A_1\) such that \(a'\) strictly dominates \(a^*\). But then

\[
v_1(a', b^*) > v_1(a^*, b^*)
\]

and \(a^*\) is not a best response for Player 1 to \(b^*\). This contradicts our assumption that \((a^*, b^*)\) is a Nash equilibrium. \(\square\)

The converse fails. For example, in *Battle of the Sexes*, \((B, S)\) and \((S, B)\) are IESDS-equilibria that are not Nash equilibria. Similarly, in *Heads/Tails* every strategy pair is an IESDS-equilibrium but no strategy pair is a Nash equilibrium.

On the other hand, the converse is true in the special case when the IESDS equilibrium is unique.
Proposition 3 If \((a^*, b^*)\) is the unique IESDS equilibrium, then \((a^*, b^*)\) is a Nash equilibrium.

Proof For purposes of contradiction suppose \(a^* \notin BR_1(b^*)\)–the other case is similar. Let
\[
X = \{a \in A_1 : v_1(a, b^*) > v_1(a^*, b^*)\}.
\]
If \(a^*\) is not a best response to \(b^*\), then \(X\) is non-empty. Since \(X\) is finite, we can find \(a' \in X\) such that \(v(a', b^*)\) is maximal.

Since \((a^*, b^*)\) is the unique strategy profile surviving, we must eliminate \(a'\) at some stage. But \(v_1(a', b^*)\) is maximal. Thus at no stage will be find a strategy strictly dominating \(a'\) and \(a'\) will never be eliminated, a contradiction. \(\square\)

Corollary 4 If there is a unique IESDS equilibrium, it is also the unique pure strategy Nash equilibrium.

Proof By Proposition 3 the unique IESDS equilibrium is a Nash equilibrium. By Proposition 2, if there was a second Nash equilibrium it would also be an IESDS equilibrium. \(\square\)

Corollary 5 If there is a strictly dominant strategy equilibrium, it is the unique Nash equilibrium.

Proof If \((a^*, b^*)\) is a strictly dominant strategy equilibrium, then in the IESDS process at stage 1 would eliminate all strategies except \(a^*\) and \(b^*\), so \((a^*, b^*)\) is the unique IESDS-equilibrium and hence the unique Nash-equilibrium. \(\square\)