Evolutionarily Stable Strategies

Consider the game

<table>
<thead>
<tr>
<th></th>
<th>Hawk</th>
<th>Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawk</td>
<td>-2,-2</td>
<td>2,0</td>
</tr>
<tr>
<td>Dove</td>
<td>0,2</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Let $\sigma_p$ be the strategy: play Hawk with probability $p$ and Dove with probability $(1-p)$.

Recall that $F(p,q)$ is the expected payoff to Player 1 when Player uses $\sigma_p$ and Player 2 uses $\sigma_q$.

**Definition 0.1** We say that $\sigma_p$ is an *Evolutionarily Stable Strategy* (ESS) if either:

i) $F(p,p) \geq F(q,p)$ for all $q \neq p$, or

ii) $F(p,p) = F(q,p)$ and $F(p,q) > F(q,q)$ for all $q \neq p$.

In case i) we say $p$ is a strong ESS and in case ii) it is a mild ESS.

Remarks:

- If $p$ is an ESS, then $F(p,p) \geq F(q,p)$ for all $q$. Thus $(\sigma_p, \sigma_p)$ must be a symmetric Nash equilibrium.

- Any strong ESS, must be a pure strategy equilibrium.

- A *strict* pure strategy symmetric equilibrium is a strong ESS. [Here strict means that if Player 1 changes her move while Player 2 does not, Player 1’s payoff will go down.]

In the Hawk–Dove game there is a unique symmetric equilibrium where each player uses the mixed strategy $\sigma_{1/3}$, i.e., play *H* with probability $1/3$. We show this is a mild ESS.

\[
F\left(\frac{1}{3}, q\right) = \frac{1}{3}[q(-2) + (1 - q)(2)] + \frac{2}{3}[q(0) + (1 - q)(1)] = \frac{4}{3} - q
\]

\[
F(q,q) = q[q(-2) + (1 - q)(2)] + (1 - q)[q(0) + (1 - q)(1)] = 1 - 4q - 3q^2
\]

We need $F(1/3, q) > F(q, q)$ for all $q \neq p$.

\[
F(1/3, q) > F(q, q)
\]

\[
\frac{4}{3} - 2q > 1 - 4q - 3q^2
\]

\[
3q^2 - 2q - \frac{1}{3} > 0
\]

\[
9q^2 - 6q - 1 > 0
\]

\[
(3q - 1)^2 > 0
\]

Which is always true as long as $q \neq 1/3$.

Thus we have a mild ESS when $p = 1/3$. In other words, it is evolutionarily stable to have a population of $1/3$ Hawks and $2/3$ Doves.
Consider next the game

<table>
<thead>
<tr>
<th></th>
<th>Slow</th>
<th>Fast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow</td>
<td>6,6</td>
<td>0,2</td>
</tr>
<tr>
<td>Fast</td>
<td>2,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

In this case both (Fast, Fast) and (Slow, Slow) are symmetric pure strategy equilibria and both are strict equilibria thus both are strong ESS.

There is a third symmetric equilibria where both players are Slow with probability \( p = \frac{1}{5} \). Since this is a mixed strategy equilibrium \( F(1/5, 1/5) = F(q, 1/5) \) for all \( q \), so it can not be a strong ESS. Is it a mild ESS?

\[
F(\frac{1}{5}, q) = \frac{1}{5}q(6) + (1 - q)(0) + \frac{4}{5}q(2) + (1 - q)(1) = \frac{7}{5}q + 1
\]

\[
F(q, q) = q[q(6) + (1 - q)(0)] + (1 - q)[q(2) + (1 - q)(1)] = 1 + q + 4q^2
\]

We need \( F(1/5, q) > F(q, q) \) for all \( q \neq p \), i.e., we need

\[
\frac{7}{5}q + 1 > 1 + q + 4q^2
\]

for all \( q \neq 1/5 \), but it easy to see this is false if, say \( q = 1 \). Then the left hand side is \( 12/5 \) and the right hand side is 6. Thus 1/5 is not an evolutionarily stable solution.

Thus in an evolutionarily stable population either everyone is Slow or everyone is Fast.

Finally consider the game

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>0,0</td>
</tr>
</tbody>
</table>

(A,A) is a symmetric strict Nash equilibrium and hence a strong ESS.

(B,B) is a Nash symmetric equilibrium, but \( F(B, A) = 0 \) while \( F(A, A) = 2 \) so \( F(B, A) \neq F(A, A) \), so B is not evolutionarily stable.

There are no mixed strategy equilibria in this game (if there is any positive probability your opponent plays A you should play A, in which case your opponent should play A). Thus (A,A) is the only ESS.