6.2.5 a) \( f_n(x) = 1 \) if \( |x| \geq 1/n \) and \( f_n(x) = n|x| \) if \( |x| < 1/n \).

For any \( x \neq 0 \), for all \( n > 1/|x| \) we have \( f_n(x) = 1 \) so \( f_0(x) = \lim f_n(x) = 1 \). If \( x = 0 \) then \( f_n(x) = 0 \) so \( f_0(0) = \lim f_n(0) = 0 \).

The limiting function \( f_0 \) is not continuous so the convergence is not uniform.
b) Let \( f_n(x) = 1/x \) for \( |x| \geq 1/n \) and \( f_n(x) = n^2x \) for \( |x| < 1/n \). Then \( f_0(x) = \lim f_n(x) = 1/x \) for \( x \neq 0 \) and \( f_0(0) = 0 \). The limit function is unbounded.

6.2.7 Suppose \( f_n \) converges uniformly to \( f \) on \( A \) and each \( f_n \) is uniformly continuous.

Given \( \epsilon > 0 \) find \( N \) such that \( |f_N(x) - f(x)| < \epsilon/3 \) (Uniform convergence). Now find a \( \delta > 0 \) so that \( |x - y| < \delta \) implies \( |f_N(x) - f_N(y)| < \epsilon/3 \). (uniform continuity) Then for \( |x - y| < \delta \)

\[
|f(x) - f(y)| = |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\
|f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\]

6.2.8
a) False. \( f_n(x) = x^n \) on \([0,1] \). The limit is \( f_0(x) = 1 \) for \( x < 1 \) and \( f_0(1) = 1 \).
b) True. Suppose \( |g| \leq M \). Given \( \epsilon > 0 \) choose \( N \) large enough so that for \( n \geq N \) that for all \( x \) we have

\[
|f(x) - f_n(x)| \leq \epsilon/M.
\]

Then

\[
|g(x)f(x) - g(x)f_n(x)| = |g(x)||f(x) - f_n(x)| \leq M|f(x)| = \epsilon.
\]

c) True. Let \( \epsilon = 1 \). Then by uniform convergence there is \( N \) such that \( |f_N(x) - f(x)| < 1 \) for all \( x \) which implies that \( |f(x)| \leq |f_N(x)| + 1 \). Now if \( |f_N(x)| \) is bounded by \( M \) then \( |f(x)| \leq M + 1 \).
d) True Given \( \epsilon > 0 \) find \( N_1 \) that works for all \( x \in A \) and \( N_2 \) that works for all \( x \in B \). Let \( N = \max(N_1, N_2) \).
e), f) Both true. For \( x < y \) we have \( f_n(x) \leq f_n(y) \) for all \( n \). Take limits as \( n \to \infty \). This gives \( f(x) \leq f(y) \).

6.2.10 Let \( f_n(x) = f(x + 1/n) \).
a) Given \( \epsilon > 0 \) there exists \( \delta > 0 \) so that \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \epsilon \). Choose \( N > 1/\delta \). Then for \( n \geq N \) we have \( 1/n < \delta \) and so \( |x + 1/n - x| < \delta \) which implies that

\[
|f(x + 1/n) - f(x)| < \epsilon.
\]
b) \( f(x) = x^2 \) is not uniformly continuous on the real line. To remind

\[
|x^2 - y^2| = |x - y||x + y|.
\]

Now let \( \epsilon = 1 \). For any \( \delta \) we can choose \( x \) so that

\[
(2x + \delta/2)\delta/2 > 1.
\]

Now let \( y = x + \delta/2 \). Then \( |x - y| = \delta/2 < \delta \) but

\[
|x - y||x + y| = (2x + \delta/2)\delta/2 > 1.
\]

The same idea works here. For any \( n \) we can find \( x \) so that

\[
|(x + 1/n)^2 - x^2| = |(2x + 1/n)/n| > 1
\]

and this says that \( f_n(x) \) does not converge uniformly to \( f(x) \).