Solutions to HW 5

Chapter 7:

1. (a) It equals $f_{2n}$. It is true for $n = 1$. Now the induction step:

$$f_1 + f_3 + \cdots + f_{2n+1} = (f_1 + f_3 + \cdots + f_{2n-1}) + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2} = f_{2(n+1)}.$$  

(b) It equals $f_{2n+1} - 1$. It is true for $n = 0$. Now the induction step:

$$f_0 + f_2 + \cdots + f_{2n+2} = (f_0 + f_2 + \cdots + f_{2n}) + f_{2n+2} = f_{2n+1} - 1 + f_{2n+2} = f_{2n+3} - 1.$$  

(c) When $n$ is odd the sum is $-(f_{n-1} + 1)$ and when $n$ is even the sum is $f_{n-1} - 1$. The cases $n = 0$, and $n = 1$ hold. Now first suppose that $n \geq 2$ is even. Then $n - 1$ is odd so by induction

$$f_0 - f_1 + \cdots + f_n = (f_0 - f_1 + \cdots - f_{n-1}) + f_n = -f_{n-2} - 1 + f_n = f_{n-1} - 1.$$  

On the other hand, if $n \geq 3$ is odd, then $n - 1$ is even and

$$f_0 - f_1 + \cdots - f_n = (f_0 - f_1 + \cdots + f_{n-1}) - f_n = f_{n-2} - 1 - f_n = -f_{n-1} - 1.$$  

(d) The sum of the squares of the first $n$ fibonacci numbers is $f_n f_{n+1}$. The induction step follows since

$$f_0^2 + \cdots + f_n^2 = (f_0^2 + \cdots + f_{n-1}^2) + f_n^2 = f_n(f_{n-1} + f_n) = f_n f_{n+1}.$$  

5. Using the fibonacci recurrence repeatedly, one obtains $f_{n+8} = 21f_{n+2} + 13f_n$. Thus if 7 divides $f_n$, then 7 also divides $f_{n+8}$. Since 7 divides $f_0$, we conclude that 7 divides $f_n$ when $n$ is a multiple of 8. On the other hand, if 7 divides $f_{n+8}$, then 7 divides $13f_n$, but $(7, 13) = 1$, therefore 7 divides $f_n$. Clearly 7 does not divide $f_n$ for $1 \leq n \leq 7$, therefore we conclude that 7 divides $f_n$ if and only if $n$ is a multiple of 8.

8. If the last square is red, then the second last last square must be blue, and there are $h_{n-2}$ ways to color the first $n - 2$ squares. If the last square is blue, then there is no restriction on the second last square, and there are $h_{n-1}$ ways to color the remaining squares. Thus $h_n = h_{n-1} + h_{n-2}$. Clearly $h_1 = 2$ and $h_2 = 3$, from which we conclude that $h_0 = 1$. Thus $h_0 = f_2$, and $h_n = f_{n+2}$, where $f_i$ is given in (7.11) page 198.

9. If the last square is red, then the second last last square must be blue or white, and there are $h_{n-2}$ ways to color the first $n - 2$ squares for each of these two possibilities. If the last square is blue (or white), then there is no restriction on the second last square, and there are $h_{n-1}$ ways to color the remaining squares. Thus $h_n = 2h_{n-1} + 2h_{n-2}$. Clearly $h_1 = 3$ and $h_2 = 8$, from which we conclude that $h_0 = 1$. The characteristic equation for the recurrence is $x^2 - 2x + 2$, from which we get $h_n = A(1 + \sqrt{3})^n + B(1 - \sqrt{3})^n$. The initial values yield $A = (\sqrt{3} + 2)/2\sqrt{3}$ and $B = (\sqrt{3} - 2)/2\sqrt{3}$.  

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16. How many positive integer solutions to \( a + b + c + d = n \) where \( a \in \{0, 1, 2\}, b \in \{0, 2, 4, 6\}, c \in \{0, 2, 4, 6, \ldots\}, d \in \{1, 2, 3, \ldots\} \). The answer is the coefficient of \( x^n \) in the generating function given.

17. \( g(x) = (1 + x^2 + \cdots)(1 + x + x^2)(1 + x^3 + x^6 + \cdots)(1 + x). \) This simplifies to \( (1-x)^{-2} = \sum (n+1)x^n \). Hence the answer is \( n+1 \).

21. Let \( C \) be a convex \((n+2)\)-gon with vertices \( x_1, \ldots, x_{n+2} \). Removing \( x_1 \) yields an \((n+1)\)-gon \( C' \) with \( h_{n-1} \) regions. Putting \( x_1 \) back yields \( n \) new regions formed by diagonals containing \( x_1 \) and the diagonal \( x_2x_{n+2} \). However, the diagonals from \( x_1 \) cut certain old regions into two parts, thus creating new regions. Thus \( h_n = h_{n-1} + n + R \), where \( R \) is the number of regions in \( C' \) that are cut by a diagonal containing \( x_1 \). There are two ways to compute \( R \). We may bijectively map each region in \( R \) to a set of three vertices in \( C' \) as follows: Let the diagonal \( x_1x_i \) first encounter the region in the diagonal \( x_jx_k \). Then we associate the three points \( \{x_i, x_j, x_k\} \). It is easy to check that this mapping is bijective, so \( R = \binom{n+1}{3} \). On the other hand, for fixed \( i \), the regions in \( R \) cut by \( x_1x_i \) correspond to the number of diagonals \( x_jx_k \), where \( j < i < k \), so \( R = \sum_{i=3}^{n+1} (i-2)(n+2-i) = \binom{n+1}{3} \). Using generating functions, we obtain \( g(x) = x^2/(1-x)^5 + x/(1-x)^3 \). This gives \( h_n = \binom{n+2}{4} + \binom{n+1}{2} \) for \( n \geq 2 \), in class: \((1+x)^n\).

26. The exponential generating function is

\[
([e^x + e^{-x}]/2) e^{2x} = (e^{4x} + 2e^{2x} + 1)/4.
\]

The coefficient of \( x^n/n! \) is \( 4^{n-1} + 2^{n-1} \).

33. The characteristic equation is \( x^3 - x^2 - 9x + 9 \) which has roots \( 1, \pm 3 \). Thus \( h_n = A + B3^n + C(-3)^n \). The initial values give the equations \( A + B + C = 0, A + 3B - 3C = 1, A + 9B + 9C = 2 \), and we obtain \( A = -1/4, B = 1/3, C = -1/12 \).

34. The characteristic equation is \( x^4 - 8x + 16 \), so the general solution is \((A + Bn)4^n\). The initial values give \( A = -1, B = 1 \).

42. The particular solution is \( Bn \), since \( 4 \) is a root of the characteristic equation of multiplicity \( 1 \). Running it through the equation gives \( B = 1 \). Thus \( h_n = (A + n)4^n \). The initial values give \( A = 3 \).

47. The particular solution is \( C + Dn \), and running it through the equation yields \( C = 13, D = 3 \). The general solution to the nonhomogeneous part is \((A + Bn)2^n \). Thus \( h_n = (A + Bn)2^n + 13 + 3n \). Using the initial values gives \( A = -12, B = 5 \).