Solutions to Homework #2:

4) Find an infinite counterexample to the statement of the marriage theorem
Solution: Let \( X = Z^+ \cup \{a\} \) and \( Y = Z^+ \) and join \( x \in X \) to \( y \in Y \) iff \( x = y \) or \( x = a \). Then Halls conditions clearly holds. On the other hand, a matching saturating \( X \) must saturate \( Z^+ \subset X \), and since these vertices have degree 1, it cannot saturate \( a \). Hence there is no matching saturating \( X \).

5) Let \( k \) be an integer. Show that any two partitions of a finite set into \( k \)-sets admit a common choice of representatives.
Solution: Let \( A_1, \ldots, A_m \) and \( B_1, \ldots, B_m \) be the two partitions. Form a bipartite graph with parts \( \{A_i\} \) and \( \{B_i\} \), and join \( A_i \) to \( B_j \) if they have an element in common. For any given collection of \( t \) \( A_i \)'s, the number of elements in their union is \( tk \), so the number of \( B_j \)'s covering these \( tk \) elements is at least \( t \). Hence the number of neighbors of these \( t \) \( A_i \)'s in the bipartite graph is at least \( t \), so Halls condition holds. By Halls theorem, we have a perfect matching. Each edge of this matching corresponds to an element of the ground set, and no two of these elements are the same since the \( A_i \)'s form a partition. Hence this matching gives a CSDR.

6) Let \( A \) be a finite set with subsets \( A_1, \ldots, A_n \) and let \( d_1, \ldots, d_n \in N \). Show that there are disjoint subsets \( D_k \subset A_k \) with \( |D_k| = d_k \) for all \( k \leq n \), if and only if \( |\cup_{i \in I} A_i| \geq \sum_{i \in I} d_i \) for all \( I \subset [n] \).
Solution: The condition is clearly necessary. Form a bipartite graph \( B \) with parts \( X, A \), where \( X = a_i^j \), for \( i \in [n] \) and \( j \in [d_i] \). Join \( a_i^j \in X \) to \( s \in A \) if \( s \in A_i \). Then the given condition implies Halls condition in \( B \), so \( B \) has a matching saturating all of \( X \). The construction of \( B \) implies that we obtains the sets required in the problem.

13) Show that a graph \( G \) contains \( k \) independent edges if and only if \( q(G - S) \leq |S| + |V(G)| - 2k \) for all sets \( S \subset V(G) \).
Solution: Let \( n = |V(G)| \) and form the graph \( G \) by adding \( n - 2k \) new vertices each adjacent to all vertices of \( G \). Then the condition of the problem corresponds to Tutte’s condition on \( G' \), so by Tutte’s theorem, \( G' \) has a perfect matching \( M \). The number vertices of \( V(G) \) that are matched to some other vertex of \( V(G) \) is at least \( n - (n - 2k) = 2k \), so we have at least \( k \) edges of \( M \) that lie entirely in \( G \).

17) Does there exist a function \( g(k) \) so that every multigraph with minimum degree at least \( 3 \) and at least \( g(k) \) vertices contains \( k \) disjoint cycles?
Solution: No, let \( W_n \) be the graph obtained from \( C_n \) by adding a new vertex adjacent to all vertices of \( C_n \) - this is sometimes called the wheel. Then \( W_n \) has \( n + 1 \) vertices, minimum degree \( 3 \) and no two disjoint cycles.

18) Prove that the vertices of a graph \( G \) can be covered by at most \( \alpha(G) \) disjoint subgraphs each isomorphic to a cycle, \( K_2 \), or \( K_1 \).
Solution: We can proceed by induction on \( \alpha(G) \). The claim clearly holds for \( \alpha(G) = 1 \). Take a longest path \( P \) in \( G \) with endpoints \( u, v \). If \( P \) has no edges, then the result holds trivially, so assume there is at least one edge in \( P \). All edges incident with \( u \) are on \( P \). Hence either \( d_G(u) = 1 \) or there is a cycle \( C \) such that \( u \in V(C) \subset V(P) \) and \( u \) has no
edges to $G - C$ (by picking the furthest neighbor of $u$ on $P$). Let $C' = C$ or the edge incident to $u$ if $d_G(u) = 1$. Any independent set of $G - C'$ can be augmented by adding $u$ to obtain an independent set in $G$, hence $\alpha(G - C') < \alpha(G)$. By induction, we can cover $G - C'$ and then cover $G$ by adding $C'$.

21) Derive Hall’s theorem from the Gallai-Milgram theorem.

**Solution** Suppose we are given the bipartite graph $B = X, Y$ with $|N(S)| \geq |S|$ for all $S \subset X$. Form a directed graph $D$ by directing all edges of $B$ from $X$ to $Y$. Pick an independent set $I \subset V(D)$. Then $N_B(I \cap X) \subset Y - I$ so by Halls condition, $|Y - I| \geq |I \cap X|$. Then

$$|I| = |I \cap X| + |I \cap Y| \leq |Y - I| + |I \cap Y| = |Y|.$$

So by Gallai-Milgram, $D$ can be covered by at most $|Y|$ directed paths. Each of these paths must have an endpoint in $Y$, so $X$ is covered by paths of length at least 1. These paths provide a matching saturating all of $X$. 