• Show your work on all questions. Incorrect answers with sufficient work may get partial credit, and correct answers with insufficient work may not get full credit.

• Calculators are not allowed (they won’t help anyway).

• You are responsible for upholding UIC’s standard for academic integrity. This includes protecting your work from the eyes of other students.

• Turn off your cell phone before the exam begins.

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<th>Problem</th>
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How was this exam?  
(a) Very hard  
(b) Hard  
(c) OK  
(d) Easy  
(e) Very easy
Problem 1. (20/20 points) Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(0, \theta)$, where $\theta > 0$ is the variance. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

1. (6/6 points) Show/argue that $N(0, \theta)$ has the monotone likelihood ratio (MLR) property in a statistic $T$, and identify $T$.

2. (7/7 points) Using the MLR property above, argue that the uniformly most powerful test is of the form: Reject $H_0$ iff $T \geq k$.

3. (7/7 points) Find $k$ above such that the test has size $\alpha$. 
Problem 2. (20/20 points) Let \( X_1, \ldots, X_n \overset{\text{iid}}{\sim} f_\theta(x) \), with \( f_\theta(x) = e^{-(x-\theta)}I_{[\theta, \infty)}(x) \), and consider testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta \neq \theta_0 \).

1. (10/10 points) The likelihood ratio test rejects \( H_0 \) iff \( L(\theta_0)/L(X_{(1)}) \leq k \). Show that this is equivalent to a test: Reject \( H_0 \) iff \( X_{(1)} - \theta_0 < 0 \) or \( X_{(1)} - \theta_0 \geq k' \).

2. (10/10 points) Find \( k' \) so that the test has size \( \alpha \). (Hint: Set \( Y_i = X_i - \theta_0 \) for \( i = 1, \ldots, n \); then \( Y_{(1)} = X_{(1)} - \theta_0 \) and \( Y_1, \ldots, Y_n \) are iid \( \text{Exp}(1) \) when \( \theta = \theta_0 \).)
Problem 3. (20/20 points) Suppose that $X_1, \ldots, X_n$ are independent (but not iid) with $X_i \sim N(\theta t_i, 1)$, $i = 1, \ldots, n$, where $t_1, \ldots, t_n$ are known constants.

1. (10/10 points) Find the maximum likelihood estimator $\hat{\theta}$ of $\theta$. (Hint: Independence means the likelihood function is still the product of the individual PDFs.)

2. (10/5 points) Find the mean and variance of $\hat{\theta}$. Is $\hat{\theta}$ unbiased?

3. (0/5 points) Suppose the $t_i$'s satisfy $\sum_{i=1}^{\infty} t_i^2 = \infty$. Show that $\hat{\theta}$ is consistent. (Hint: Use Chebyshev’s inequality to show that $\lim_{n \to \infty} P_\theta(\left| \hat{\theta} - \theta \right| > \varepsilon) = 0$ for any $\varepsilon > 0$.)
Problem 4. (20/20 points) In a Bayesian setup, suppose that the prior distribution is $\Theta \sim \text{Gamma}(a, b)$ and, given $\Theta = \theta$, the model is $X \sim \text{Pois}(\theta)$. In this case, write the $\text{Gamma}(a, b)$ PDF as $\pi(\theta) \propto \theta^{a-1}e^{-b\theta}$, so that the prior mean is $E(\Theta) = a/b$.

1. (5/5 points) Show that the posterior distribution for $\Theta$, given $X = x$, is $\text{Gamma}(a', b')$, where $a' = a + x$ and $b' = b + 1$.

2. (5/5 points) Suppose $a = b = 1$. What’s the posterior mean $E(\Theta | x)$?

3. (5/5 points) Consider the MVUE $\hat{\theta}_1 = X$ and the posterior mean $\hat{\theta}_2 = E(\Theta | X)$ above. Find the respective mean-square errors, $\text{MSE}_1(\theta)$ and $\text{MSE}_2(\theta)$.

4. (5/5 points) Which of $\text{MSE}_1(1)$ and $\text{MSE}_2(1)$ is larger? Use the particular choice of prior distribution for $\theta$ to make a case for why having small mean-square error for $\theta$ near 1 might be important.
Problem 5. (20/15 points) Undergraduate students must answer any two of the four questions; graduate students must answer any three.

1. There is a theorem that says a Bayes posterior mean \( \hat{\theta} = E(\Theta \mid X) \) cannot be an unbiased estimator of \( \theta \). Does this mean that Bayes estimators are bad? Explain your answer. (Hint: Consider the bias–variance tradeoff.)

2. Use the Neyman–Fisher factorization theorem to argue that the Bayesian posterior distribution for \( \Theta \) depends on data only through a sufficient statistic.

3. For testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta > \theta_0 \), suppose the p-value is given by \( \text{pval}(t) = P_{\theta_0}(T \geq t) \), where \( t \) is the observed value of the test statistic \( T \). Explain why a small p-value is indication that \( H_0 \) may be false.

4. Explain how Wilk’s theorem is used in the context of hypothesis testing and why it’s important.
Problem 6. (0/5 points) Graduate students must answer one of the two questions. Refer back to the situation in Problem 1.

1. Let $\text{pow}(\theta)$ denote the power function for the size-0.05 test in Problem 1(c). Show that, for any $\theta > \theta_0$, $\text{pow}(\theta) \to 1$ as $n \to \infty$. (Hint: Central Limit Theorem.)

2. Let $T = \left(\frac{1}{\theta_0}\right) \sum_{i=1}^{n} X_i^2$ and write $t$ for the value of $T$ for the observed sample. Then the p-value for the test is $p\text{val}(t) = P_{\theta_0}(T \geq t)$. The null distribution of $T$ is known, but pretend that we don’t know it. Explain how you could numerically evaluate the p-value using Monte Carlo.