Rational Equivariant Forms

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This is joint work with Abdellah Sebbar.
Let us fix some notation:

\[ \mathcal{H} := \{ z \in \mathbb{C}; \Im(z) > 0 \} , \quad \mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) , \]

\[ \text{SL}_2(\mathbb{Z}) := \text{the modular group}, \]

\[ \alpha \cdot z := \frac{az + b}{cz + d}, \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \quad z \in \mathbb{C}. \]

\( \Gamma \) is a subgroup of \( \text{SL}_2(\mathbb{Z}) \) of finite index, which we call a modular subgroup.

(We would like to mention that all what we present here, in fact, holds for any discrete subgroup of \( \text{SL}_2(\mathbb{R}) \)).
Preliminaries: The Schwarz derivative:

For a meromorphic function on a domain (open and connected) of \( \mathbb{C} \), the Schwarz derivative, denoted \( \{ f, z \} \), is defined by

\[
\{ f, z \} = 2 \left( \frac{f''}{f'} \right)' - \left( \frac{f''}{f'} \right)^2 \tag{1}
\]

\[
= \frac{2f'f''' - 3f''^2}{f'^2}.
\]

It satisfies the following rules.

- **Chain rule:** If \( w \) is a function of \( z \) then
  \[
  \{ f, z \} = (dw/dz)^2 \{ f, w \} + \{ w, z \}.
  \]

- Consequently, for \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{C}) \), we have
  \[
  \{ w, z \} = \frac{\det \alpha}{(cz + d)^4} \{ w, \alpha \cdot z \}. \tag{2}
  \]
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(2)
Preliminaries: The Schwarz derivative:

- \( \{f, z\} = 0 \) if and only if \( f \) is a linear fractional transform of \( z \).
- \( \{w_1, z\} = \{w_2, z\} \) if and only if each function \( (w_i) \) is a linear fraction of the other.
- Inversion formula: If \( w'(z_0) \neq 0 \) for some point \( z_0 \), then in a neighborhood of \( z_0 \),
  \[
  \{z, w\} = -(dz/dw)^2\{w, z\}.
  \]

Lastly, given a meromorphic function \( f \) on a domain \( D \) of \( \mathbb{C} \), then one deduces the following proposition:

**Proposition**

The Schwarz derivative \( \{f, z\} \) of \( f \) has a double pole at the critical points of \( f \) and is holomorphic elsewhere including at simple poles of \( f \).
This is due to J. McKay and A. Sebbar.
Suppose now that $f$ is a modular function for some modular subgroup $\Gamma$.

**Proposition (M-S)**

i. If $f$ is a modular function for $\Gamma$ then $\{f, z\}$ is a (meromorphic) weight 4 modular form for $\Gamma$.

ii. If in addition $\Gamma$ is of genus 0 and $f$ is a Hauptmodul for $\Gamma$, then $\{f, z\}$ is weight 4 (holomorphic if $\Gamma$ is torsion free) modular form for the normalizer of $\Gamma$ inside $SL_2(\mathbb{R})$.

What about the converse?
In other words, given meromorphic function $f$ on $\mathbb{H}$ such that $\{f, z\}$ is a weight 4 modular form on a modular subgroup $\Gamma$, what can be said about the invariance group $G_f$ of $f$. 
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What about the converse?

In other words, given meromorphic function \( f \) on \( \mathbb{H} \) such that \( \{f, z\} \) is a weight 4 modular form on a modular subgroup \( \Gamma \), what can be said about the invariance group \( G_f \) of \( f \).
In fact, using properties of the Schwarz derivative, we have for \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \)

\[
\{ f(\alpha \cdot z), \alpha \cdot z \} = (cz + d)^4 \{ f(z), z \} \quad \text{(modularity)}
\]

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\]

Hence

\[
\{ f(\alpha \cdot z), z \} = \{ f(z), z \}
\]

and therefore

\[
f(\alpha \cdot z) = \Phi_\alpha \cdot f(z), \quad \text{for some } \Phi_\alpha \in \text{GL}_2(\mathbb{C}).
\]
In particular, we have

\[ \Phi : \Gamma \longrightarrow \text{GL}_2(\mathbb{C}) \]

\[ \alpha \mapsto \Phi\alpha \]

is a group homomorphism. Moreover, \( G_f = \text{Ker} \Phi \).

**Theorem (S)**

If \( f \) is as above and is such that \( f(z + n) = f(z) \), \( n \in \{1, 2, 3, 4, 5\} \), then \( f \) is a modular function for \( \Gamma(n) = G_f \).

There are some cases where \( G_f \), for instance \( \log(f) \), \( f \) a Hauptmodul of \( \Gamma(n) \) with \( n \) as above, is no larger than \( < T^m > \), for some \( m \) \( (T = (1 1 \atop 0 1)) \).

**Question:** when \( G_f \) is trivial?
This is the case if \( \Phi \) is injective.
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For the rest of this talk, $\Phi$ is the natural injection of $\Gamma$ into $\text{GL}_2(\mathbb{C})$. Hence, our functions are of the following type.

**Definition**

A meromorphic function $h$ on $\mathcal{H}$ is called an equivariant function for $\Gamma$ if it satisfies

$$h(\alpha \cdot z) = \alpha \cdot h(z), \quad \text{for all } \alpha \in \Gamma.$$  

A first, obvious, example is $f(z) = z$.

A larger class comes from

**Proposition**  

Let $f$ be a weight $k$, $k \in \mathbb{Z}$, modular form for $\Gamma$. Then the function

$$h_f(z) = z + \frac{kf(z)}{f'(z)} \quad (\star)$$

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Equivariant "functions": first examples

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The general definition

We define a "slash operator" on equivariant functions via the following action of $\text{SL}_2(\mathbb{R})$ on meromorphic functions on $\mathcal{H}$. For a meromorphic function $f$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, we let

$$f\|_{[\gamma]}(z) = j_\gamma(z)^{-2}f(\gamma \cdot z) - cj_\gamma(z)^{-1}, \quad j_\gamma(z) = cz + d.$$ 

For a meromorphic function $h$ on $\mathcal{H}$, set $H(z) = (h(z) - z)^{-1}$. Then we have

**Proposition**

The function $h$ is an equivariant function for $\Gamma$ if and only if $H\|_{[\gamma]}(z) = H(z)$ for all $\gamma \in \Gamma$. Furthermore, $H\|_{[-\gamma]}(z) = H\|_{[\gamma]}(z)$ if $-1_2 \in \Gamma$. 
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An equivariant function for \( \Gamma \) is called an **equivariant form** for \( \Gamma \) if it satisfies

1. \( h \) is meromorphic on \( \mathcal{H} \);
2. \( h \) is meromorphic at the cusps, meaning that the function \( H||[\gamma](z) \) is meromorphic at infinity for all \( \gamma \in \text{SL}_2(\mathbb{Z}) \).

**Example**

For \( f \in M_k^m(\Gamma), \ k \neq 0 \), the equivariant function \( h_f = z + kf/f' \) (as in (\( \star \))) is therefore an equivariant form.

**Proposition**

Suppose that \( \Gamma_1 \) and \( \Gamma_2 \) are conjugate subgroups of \( \text{SL}_2(\mathbb{Z}) \), say \( \Gamma_1 = \alpha \Gamma_2 \alpha^{-1} \), for \( \alpha \in \text{GL}_2^+(\mathbb{Q}) \). Then if \( h_1 \) is an equivariant form on \( \Gamma_1 \), the function \( h_2(z) = \alpha^{-1} \circ h_1 \circ \alpha(z) \) is an equivariant form on \( \Gamma_2 \).
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Rational Equivariant forms

**Definition**

An equivariant form is called a **rational equivariant form** if it arises from a modular form (of weight $k \neq 0$).

**Proposition**

For $c \in \mathbb{C}$ and $n \in \mathbb{Z}^\times$, we have

$$h f^n = h cf = h f = z + k f / f', f \in M^m_k(\Gamma).$$

**Proposition**

For conjugate subgroups $\Gamma_1, \Gamma_2$, if $h_1(z) = z + k f(z) / f'(z)$, $f \in M^m_k(\Gamma_1)$, is a rational equivariant form then so is 

$$h_2(z) = \alpha^{-1} \circ h_1 \circ \alpha(z),$$

and we have 

$$h_2(z) = z + \frac{k(f|_{k[\alpha]})(z)}{(f|_{k[\alpha]})'(z)}.$$
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Rational Equivariant Forms

Example

A fundamental example corresponds to the modular discriminant $\Delta$

$$h_\Delta(z) = z + 12\Delta(z)/\Delta'(z) = z + 6/\pi i E_2(z).$$

Remarks

- The trivial example $h_t(z) = z$ does not fit in the above setting, however we can ”associate” it to modular functions.

- The zeros and poles of the function $H_f(z) = (h_f(z) - z)^{-1}$ are all simple and have rational residues. Indeed, if $n$ is the multiplicity of $f$ at $z_0$ (a pole or a zero), then $z_0$ is a simple pole of $H_f$ with residue $n/k$. At a cusp $s$ of $\Gamma$ and $\gamma \in \text{SL}_2(\mathbb{Z})$ such that $\gamma \cdot s = \infty$, one can see that

$$\frac{1}{2i\pi} \lim_{z \to i\infty} H_f||[\gamma^{-1}](z) = \frac{n}{k l_s} \in \mathbb{Q},$$

where $n$ is the order of infinity in the $q_s$-expansion of $f|_k[\gamma^{-1}](z)$ and $l_s$ is the cusp width at $s$ of $\Gamma$. This justifies the use of ”rational”.
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**Question:** When an equivariant form is a rational equivariant form?

The following proves that the conditions of the remark are in fact also sufficient.

**Theorem**

Let $\Gamma$ be a modular subgroup and let $h$ be an equivariant form for $\Gamma$. Then $h$ is rational if and only if

1. The poles of $H$ in $\mathcal{H}$ are all simple with rational residues.
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To prove this one has to establish the following lemmas.
**Lemma**

Let $\Gamma$ be a modular subgroup, and let $h$ be an equivariant form for $\Gamma$. Then $H$ has only a finite number of poles in the closure of a fundamental domain of $\Gamma$.

**Lemma**

Suppose that $h$ is equivariant for a modular subgroup $\Gamma$ such that $H$ has only simple poles in $\mathcal{H}$, then the set of the residues at these poles is finite.

The theorem then follows by associating to $h$ the function

$$f(z) = \exp \left( \int_{z_0}^{z} kH(u)\,du \right),$$

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In the case of genus zero subgroups

**Theorem**

Let $h$ be an equivariant form on a subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ of genus 0 such that $h(z) \neq z$ for all $z \in H$. Suppose also that, for every cusp $s$, if $\gamma \in \text{SL}_2(\mathbb{Z})$ is such that $\gamma \cdot s = \infty$, we have

$$\lim_{z \to i\infty} H||[\gamma^{-1}](z) = a_s$$

is finite and satisfies $(a_s/6) \in \pi i \mathbb{Z}$. Then

$$h(z) = h_\Delta(z) = z + \frac{6}{\pi i E_2(z)}.$$
Yes!

**Theorem**

Let $\Gamma$ be a modular subgroup and let $f$ and $g$ be modular forms of weights $k$ and $k + 2$ respectively, then

$$h(z) = z + k \frac{f(z)}{f'(z) + g(z)}$$

is an equivariant form for $\Gamma$.

A complete classification will appear in a joint work with Abdellah Sebbar, with a complete structure (an affine space) and geometric “correspondences”.
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Are there other examples?

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Proposition

If \( h \) is an equivariant form for \( \Gamma \), then \( \{h, z\} \) is a weight 4 modular form on \( \Gamma \).

In the case of rational equivariant forms, this is connected to the Cohen-Rankin bracket, which is defined for \( n \geq 0 \) with \( D^j = \frac{d^j}{dz^j} \) by

\[
[f, g]_n = \sum_{r+s=n} \binom{k+n-1}{r+s} \binom{l+n-1}{r} D^r f \ D^s g , \quad r, s \geq 0 .
\]

It is known that for \( f \in M_k^m(\Gamma) \) and \( g \in \mathcal{M}_l^m(\Gamma) \) and for every \( n \geq 0 \), the function \([f, g]_n\) belongs to \( \mathcal{M}_{k+l+2n}^m(\Gamma) \).

Proposition

Let \( f \) be a modular form of weight \( k \) on \( \Gamma \) and \( h_f \) the corresponding rational equivariant form. Then \( f'^2 h'_f \) is a weight \( 2k + 4 \) modular form on \( \Gamma \). Moreover, the poles of \( \{h_f, z\} \) are located at the zeros of the second Cohen-Rankin bracket \([f, f]_2\) of \( f \).
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It is known that for \( f \in \mathcal{M}_k^m(\Gamma) \) and \( g \in \mathcal{M}_l^m(\Gamma) \) and for every \( n \geq 0 \), the function \([f, g]_n\) belongs to \( \mathcal{M}_{k+l+2n}^m(\Gamma) \).

Proposition

Let \( f \) be a modular form of weight \( k \) on \( \Gamma \) and \( h_f \) the corresponding rational equivariant form. Then \( f^\prime h_f^\prime \) is a weight \( 2k + 4 \) modular form on \( \Gamma \). Moreover, the poles of \( \{ h_f, z \} \) are located at the zeros of the second Cohen-Rankin bracket \([f, f]_2\) of \( f \).
The Effect of the Schwarz Derivative and the Cohen-Rankin Bracket

Proposition

If \( h \) is an equivariant form for \( \Gamma \), then \( \{h, z\} \) is a weight 4 modular form on \( \Gamma \).

In the case of rational equivariant forms, this is connected to the Cohen-Rankin bracket, which is defined for \( n \geq 0 \) with \( D^j = \frac{d^j}{dz^j} \) by

\[
[f, g]_n = \sum_{r+s=n} \binom{k+n-1}{s} \binom{l+n-1}{r} D^r f D^s g , \quad r, s \geq 0 .
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It is known that for \( f \in \mathcal{M}_k^m(\Gamma) \) and \( g \in \mathcal{M}_l^m(\Gamma) \) and for every \( n \geq 0 \), the function \( [f, g]_n \) belongs to \( \mathcal{M}_{k+l+2n}^m(\Gamma) \).

Proposition

Let \( f \) be a modular form of weight \( k \) on \( \Gamma \) and \( h_f \) the corresponding rational equivariant form. Then \( f'^2 h'_f \) is a weight \( 2k + 4 \) modular form on \( \Gamma \). Moreover, the poles of \( \{h_f, z\} \) are located at the zeros of the second Cohen-Rankin bracket \( [f, f]_2 \) of \( f \).
Theorem 6

The Eisenstein series $E_2$ has infinitely many zeros in the half-strip $\mathcal{S} = \{ \tau \in \mathbb{H}, -\frac{1}{2} < \text{Re}(\tau) \leq \frac{1}{2} \}$.

The proof is as follows. One first observes that, for each $z \in \mathbb{H}$ such that $h_\Delta(z) \in \mathbb{Q}$, the $\text{SL}_2(\mathbb{Z})$-equivalence class of $z$ contains a zero of $E_2$. Fix such a point $z_0$. Then, in any neighborhood $U$ of $z_0$, the function $h_\Delta$ takes infinitely many rational values, and so any neighborhood of $z_0$ contains infinitely many points that are equivalent to a zero of $E_2$. If $U$ is sufficiently small then, each equivalence class of a point of $U$ meets $U$ in at most 3 points (about the vertices of a fundamental domain of $\text{SL}_2(\mathbb{Z})$). In fact, as $z_0$ cannot be an elliptic point, one can choose $U$ such that $U$ is in the interior of some fundamental domain for $\text{SL}_2(\mathbb{Z})$, and then the points of $U$ are two by two inequivalent.
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An Application: The zeros of $E_2$

So, $U$ contains infinitely many points in the equivalence class of distinct zeros of $E_2$.

Note that the strict monotonicity of $E_2$ on the imaginary axis provides us with the point $z_0$.

Finally, shift all the zeros to the strip $\mathcal{S}$ and notice that $E_2$ is invariant under translation and has integer coefficients.
The Cross-ratio of Equivariant Forms

Proposition

Let $h_1$, $h_2$, $h_3$, $h_4$ be four equivariant forms on a modular subgroup $\Gamma$. Define a function $f$ as the cross-ratio of these four elements

$$f = (h_1, h_2; h_3, h_4) = \frac{(h_1 - h_3)(h_2 - h_4)}{(h_2 - h_3)(h_1 - h_4)}.$$

Then, if $h_2 \neq h_3$ and $h_1 \neq h_4$, the function $f$ is a modular function on $\Gamma$.

This actually gives a parametrization of equivariant forms by modular functions.

Proposition

We have

$$1728(h_t, h_{E_6}; h_{E_4}, h_\Delta) = j.$$
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\[ f = (h_1, h_2; h_3, h_4) = \frac{(h_1 - h_3)(h_2 - h_4)}{(h_2 - h_3)(h_1 - h_4)}. \]

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\[ 1728(h_t, h_{E_6}; h_{E_4}, h_\Delta) = j. \]
Thank you for your attention!

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Special thanks to Ramin!