DETERMINACY IN THIRD ORDER ARITHMETIC

SHERWOOD J. HACHTMAN

Abstract. In recent work, Schweber [7] introduces a framework for reverse mathematics in a third order setting and investigates several natural principles of transfinite recursion. The main result of that paper is a proof, using the method of forcing, that in the context of two-person perfect information games with moves in $\mathbb{R}$, open determinacy ($\Sigma_1^{\mathbb{R}}$-DET) is not implied by clopen determinacy ($\Delta_1^{\mathbb{R}}$-DET). In this paper, we give another proof of this result by isolating a level of L witnessing this separation. We give a notion of β-absoluteness in the context of third-order arithmetic, and show that this level of L is a β-model; combining this with our previous results in [2], we show that Σ_4^0-DET, determinacy for games on ω with Σ_4^0 payoff, is sandwiched between $\Sigma_1^{\mathbb{R}}$-DET and $\Delta_1^{\mathbb{R}}$-DET in terms of β-consistency strength.

§1. Introduction. Reverse mathematics, initiated and developed by Harvey Friedman, Stephen Simpson, and many others, is the project of classifying theorems of ordinary mathematics according to their intrinsic strength (a thorough account of the subject is given in [8]). This project has been enormously fruitful in clarifying the underlying strength of theorems, classical and modern, formalizable in second order arithmetic. However, the second order setting precludes study of objects of higher type (e.g., arbitrary functions $f : \mathbb{R} \to \mathbb{R}$), and a number of frameworks have been proposed for reverse math in higher types. For example, Kohlenbach [3] develops a language and base theory RCA_0^ω to accommodate all finite types, and shows it is conservative over the second order theory RCA_0; Schweber [7] defines a theory RCA_0^3 for three types over which RCA_0^ω is conservative.

In this paper, we are interested in higher types because of their necessary use in proofs of true statements of second order arithmetic, namely, in proofs of Borel determinacy. The reverse mathematical strength of determinacy for the first few levels of the Borel hierarchy has been well-investigated ([9], [10], [11], [12], [6]). However, as Montálban and Shore [6] show, determinacy even for ω-length differences of Π_3^0 sets is not provable in Z_2, full second order arithmetic, and by the celebrated results of Friedman [1] and Martin [4], [5], determinacy for games with Σ_{n+4}^0 payoff, for $n \in \omega$, requires the existence of $\mathcal{P}^{n+1}(\omega)$, the $n+1$-st iterated Power set of ω.

In light of this, the third order framework developed by Schweber [7] seems a natural setting for investigating the strength of Σ_4^0 determinacy. In addition to defining the base theory RCA_0^3, Schweber introduces a number of natural versions of transfinite recursion principles in the third order context; he then proceeds to show that many of these are not equivalent over the base theory. In particular, he shows that Open determinacy for games played with *real-number* moves ($\Sigma^{\mathbb{R}}_1$-DET) is *strictly stronger* than Clopen determinacy ($\Delta^{\mathbb{R}}_1$-DET) . The argument given there is a technical forcing construction, and it is asked ([7] Question 5.2) whether this separation is witnessed by some level of Gödel's L, say the least satisfying "$P(\omega)$ exists + $\Delta^{\mathbb{R}}_1$-DET".

This latter question turns out to be related to our recent work [2] investigating the strength of Σ^0_4-DET, determinacy for games on ω with Σ^0_4 payoff. As mentioned above, this determinacy lies beyond the reaches of ordinary second order arithmetic, and so the calibration there is given in terms of models of set theory, viz., levels of L. For example,

THEOREM 1.1 ([2]). *Working over Π^1_1-CA_0, the determinacy of all Σ^0_4 games is equivalent to the existence of a countable ordinal θ so that $L_\theta \models$ "$\mathcal{P}(\omega)$ exists, and for any tree T of height ω, either T has an infinite branch or there is a map $\rho : T \to \mathrm{ON}$ so that $\rho(x) < \rho(y)$ whenever $x \supsetneq y$."*

If θ is the least such ordinal, then it is also the least ordinal so that every Σ^0_4 game is determined as witnessed by a strategy in $L_{\theta+1}$.

In fact, we found L_θ is a model of $\mathsf{RCA}^3_0 + \Delta^{\mathbb{R}}_1$-DET $+\neg\Sigma^{\mathbb{R}}_1$-DET, so answering Schweber's question in the affirmative; this is proved in the next section.

In light of this result, it seemed plausible that the results of [2] could be elegantly reformulated in terms of higher-order arithmetic. Indeed, the defining property of L_θ bears a resemblance to that of β-model from reverse mathematics: a structure (ω, S) (where $S \subseteq \mathcal{P}(\omega)$) in the language of second order arithmetic is a *β-model* if it satisfies all true Σ^1_1 statements in parameters from S. We wondered: Can the three-sorted structure $(\omega, (\mathbb{R})^{L_\theta}, (\omega^{\mathbb{R}})^{L_\theta})$ be characterized as a minimal β-model of some natural theory in third-order arithmetic?

We here provide such a characterization. We describe a translation from β-models in third-order arithmetic to transitive models of set theory, much in the spirit of the second order translation given in [8]. Combining these results with the theorem, we obtain: Σ^0_4-DET is equivalent over Π^1_1-CA_0 to the existence of a countably-coded β-model of *projective transfinite recursion*, or Π^1_∞-$\mathsf{TR}_{\mathbb{R}}$; as we shall see, the latter theory is the natural analogue of ATR_0 in the third-order setting, and is equivalent (modulo the existence of selection functions for $\mathbb{R}$-indexed sets of reals) to $\Delta^{\mathbb{R}}_1$-DET.

§2. Separating $\Sigma^{\mathbb{R}}_1$-DET and $\Delta^{\mathbb{R}}_1$-DET. We begin by showing that L_θ is a witness to the main separation result of Schweber [7].

THEOREM 2.1. *L_θ is a model of $\Delta^{\mathbb{R}}_1$-DET, but not of $\Sigma^{\mathbb{R}}_1$-DET.*

PROOF. Working in L_θ, suppose $T \subseteq \mathcal{P}(\omega)^{<\omega}$ is a tree with no infinite branch. We will show that the game where Players I and II alternate choosing nodes of a branch through T is determined (here a player loses if he is the first to leave T).

In [2], it is shown that L_θ is a model of the following Π_1-*Reflection Principle* (Π_1-RAP): Whenever Q is a set of reals (that is, $Q \subseteq \mathcal{P}(\omega)$) and $\varphi(Q)$ is a true Π_1 formula, there is some admissible set M so that $Q \cap M \in M$, $M \models$ "$\mathcal{P}(\omega)$ exists", and $\varphi(Q \cap M)$ holds in M.

Suppose the game on T is undetermined. This is a Π_1 statement in parameters: it states that for any strategy σ for either player, there is a finite sequence $s \in \mathcal{P}(\omega)^{<\omega}$ against which this strategy loses the game on T (note that we may use $\mathcal{P}(\omega)$ as a parameter, so the existential quantifier is bounded). By Π_1-RAP, let M be an admissible set with $\bar{T} = T \cap M \in M$ so that $M \models$ "$\mathcal{P}(\omega)$ exists and neither player wins the game on $\bar{T}$". Note that $\bar{T}$ is a wellfounded tree (in V), and by basic properties of admissible sets, we have a map $f : \bar{T} \to \mathrm{ON} \cap M$ in M so that $s \subsetneq t$ implies $f(s) > f(t)$. Working in M, we may therefore define by induction on the wellfounded relation $\supsetneq \cap (\bar{T} \times \bar{T})$ a partial function $\rho : \bar{T} \to \mathrm{ON}$ in M by

$$\rho(s) = \mu\alpha[(\forall x)(\exists y) s^\frown\langle x\rangle \in \bar{T} \to \rho(s^\frown\langle x, y\rangle) < \alpha].$$

Let us say an element in the domain of ρ is *ranked*. We claim for every $s \in \bar{T}$, either s is ranked or some real x exists with $s^\frown\langle x\rangle \in \bar{T}$ ranked. For suppose not, and let s be $\supsetneq$-minimal such. Then whenever x is such that $s^\frown\langle x\rangle \in \bar{T}$, there is some y so that $\rho(s^\frown\langle x, y\rangle)$ exists. By admissibility, we can find some α so that if $s^\frown\langle x\rangle \in \bar{T}$, then for some y, $\rho(s^\frown\langle x, y\rangle) < \alpha$.

So, either $\emptyset$ is ranked, or $\langle x\rangle$ is ranked for some x. It is easy to see that a winning strategy in the game on $\bar{T}$ (for II in the first case, I in the second) is definable from ρ. But this contradicts the fact that the game on $\bar{T}$ is not determined in M.

So $\Delta^{\mathbb{R}}_1$-DET holds in L_θ. It remains to show $\Sigma^{\mathbb{R}}_1$-DET fails. Note that if $T \in L_\theta$ is a tree on $\mathcal{P}(\omega)^{L_\theta}$, then if σ is a winning strategy (for either player) for the game on T in L_θ, σ is also winning in V (if σ is for the open player, then being a winning strategy is simply the statement that no terminal nodes are reached by σ; if σ is for the closed player, then the tree of plays in T compatible with τ is wellfounded, so is ranked in L_θ).

Note further that L_θ is not admissible, and Σ_1-projects to ω with parameter $\{\omega_1^{L_\theta}\}$; in particular, L_θ does not contain the real $\{k \mid L_\theta \models \phi_k(\omega_1^{L_\theta})\}$ (here $\langle \phi_k\rangle_{k\in\omega}$ is some standard fixed enumeration of Σ_1 formulae with one free variable). We will define an open game on $L_{\omega_1^{L_\theta}}$ so that Player II (the closed player) wins in V, but any winning strategy for II computes this theory; by what was just said, no winning strategy can belong to L_θ.

For the rest of the proof, we let ω_1 denote $\omega_1^{L_\theta}$. The game proceeds as follows: In round -1, Player I plays an integer k; Player II responds with 0 or 1, and a model M_0. In all subsequent rounds $n < \omega$, Player I plays a real x_n in $\mathcal{P}(\omega)^{L_\theta}$, and Player II responds with a pair π_n, M_{n+1}:

I	k	x_0	x_1	$\ldots$
II	$i \in \{0,1\}, M_0$	π_0, M_1	π_1, M_2	$\ldots$

Player II must maintain the following conditions, for all $n \in \omega$:

- M_n is a countable transitive model of "$\mathcal{P}(\omega)$ exists";
- $\pi_n : M_n \to M_{n+1}$ is a Σ_0-elementary embedding with $\pi_n(\omega_1^{M_n}) = \omega_1^{M_{n+1}}$;
- $x_n \in M_{n+1}$, and for all trees $T \in M_n$, $\pi_n(T)$ is either ranked or illfounded in M_{n+1};
- For all $a \in M_n$, $M_{n+1} \models (\exists\alpha)\pi_n(a) \in L_\alpha$;
- $M_n \models \phi(\omega_1^{M_n})$ if and only if $i = 1$.

Note the second condition entails that π_n has critical point $\omega_1^{M_n}$. The first player to violate a rule loses; Player II wins all infinite plays where no rules are violated.

We first claim that Player II wins this game in V. We describe a winning strategy. If I plays k, have Player II respond with 1 if and only if $\phi(\omega_1)$ holds in L_θ. If 1 was played, let $\alpha_0 < \theta$ be sufficiently large that $L_{\alpha_0} \models \phi(\omega_1)$; otherwise let $\alpha_0 = \omega_1 + \omega$. Inductively, let $\alpha_{n+1} < \theta$ be the least limit ordinal so that every wellfounded tree in L_{α_n} is ranked in $L_{\alpha_{n+1}}$. (Note such exists: the direct sum of all wellfounded trees $T \in L_{\alpha_n}$ belongs to L_{α_n+1}, since L_{α_n} has Σ_1 projectum $\omega_1^{L_\theta}$. If β is large enough that this sum is ranked in L_β, then all wellfounded trees of L_{α_n} are also ranked in L_β.)

Now let H_0 be the Σ_0-Hull of $\{L_{\omega_1}\}$ in L_{α_0} (that is, H_0 is the closure in L_{α_0} of $\{L_{\omega_1}\}$ under taking $<_L$-least witnesses to bounded existential quantifiers). Let M_0 be its transitive collapse. Inductively, having defined $H_n \subset L_{\alpha_n}$ and given a real x_n played by I, let H_{n+1} be the Σ_0-Hull of $H_n \cup \{L_{\alpha_n}, x_n\} \cup \{f \in L_{\alpha_n} \mid f \text{ is the rank function of a wellfounded tree } T \in H_n\}$ inside $L_{\alpha_{n+1}}$. Let M_{n+1} be its transitive collapse, and $\pi_{n,n+1} : M_n \to M_{n+1}$ be the map induced by the inclusion embedding. Inductively, each H_n (hence M_n) is countable and belongs to L_θ (since θ is limit). The remaining rules are clearly satisfied by the π_n, M_n. So II wins in V, as desired.

All that's left is to show that any winning strategy for II responds to k with 1 if and only if $L_\theta \models \phi_k(\omega_1)$; it follows that no winning strategy (for either player) can belong to L_θ. So suppose σ is winning for II. Let $\langle x_n \rangle_{n\in\omega}$ be an enumeration of the reals of $\mathcal{P}(\omega)^{L_\theta}$. Then σ replies with a sequence $\langle \pi_n, M_n \rangle_{n\in\omega}$ of models and embeddings; these form a directed system. Let M_ω be the direct limit, with $\pi_{n,\omega} : M_n \to \omega$ the limit embedding. Since $\operatorname{crit}(\pi_n) = \omega_1^{M_n}$ for each n, the ω_1 of M_ω is wellfounded. Moreover, by the rules of the game, M_ω is a model of $V = L+$ "all wellfounded trees are ranked", and since all reals of L_θ were played, we have $\omega_1^{M_\omega} = \omega_1^{L_\theta}$.

Now suppose towards a contradiction that σ played a truth value of $\phi(\omega_1)$ that disagreed with the truth value of $\phi(\omega_1)$ in L_θ. Then the model M_ω is illfounded; let $\mathrm{wfo}(M_\omega)$ be the supremum of its wellfounded ordinals. By the truncation lemma for models of $V = L$ (Proposition 2.5 in [2]), $L_{\mathrm{wfo}(M_\omega)}$ is admissible. But by minimality in the definition of θ, no α with $\omega_1 < \alpha \leq \theta$ can have L_α be admissible. So we must have $\mathrm{wfo}(M_\omega) > \theta$. But as remarked above, L_θ Σ_1-projects to ω, whereas $L_{\mathrm{wfo}(M_\omega)}$ is a proper end extension of L_θ with the same ω_1. This is a contradiction. ⊣

§3. β-models of third-order arithmetic. We adopt the definitions of the language and structures of third-order arithmetic introduced in [7]. We briefly recall some salient points. The language L^3 is a many-sorted first order language consisting of three sorts: s_0 corresponds to naturals, s_1 to functions $x : \omega \to \omega$, and s_2 to functionals $F : \omega^\omega \to \omega$. Non-logical symbols include the usual signature $\{+, \times, <, 0, 1\}$ for arithmetic on s_0, application operators $\cdot_0$ and $\cdot_1$, equality relations $=_0, =_1, =_2$ for their respective sorts, and binary operations $* : s_2 \times s_1 \to s_1$ and $^\frown : s_0 \times s_1 \to s_1$. The latter operations are introduced to allow for coding. Namely, under the intended interpretation,

$$k^\frown x = \langle k, x(0), x(1), x(2), \dots \rangle$$
$$F * x = \langle F(0^\frown x), F(1^\frown x), F(2^\frown x), \dots \rangle.$$

Here of course we are denoting type 1 objects $x : \omega \to \omega$ as $\langle x(0), x(1), \dots \rangle$. Note that in what follows we will adopt the convention that the first time a fresh variable appears, its type will be denoted by a superscript (x^1, F^2, etc.).

Recall that a structure $\mathcal{M} = (M_0, M_1)$ in the language of second order arithmetic is a *β-model* if M_0 is isomorphic to the standard natural numbers, and whenever $x \in M_1$ and $\phi(u)$ is a Σ^1_1 formula, we have $\mathcal{M} \models \phi(x)$ if and only if $\phi(x)$ is true (understood as a statement about the unique real corresponding to x). Equivalently, $\mathcal{M}$ is a β-model if whenever $T \subseteq \omega^{<\omega}$ is a tree coded by a real in M_1 (under some standard coding of finite sequences of naturals), we have that $\mathcal{M} \models$ "T is illfounded" whenever T is illfounded. For simplicity's sake, we use the latter characterization to define a notion of β-model in the third-order context.

Fix a coding $\langle \cdot \rangle : \mathbb{R}^{\leq \omega} \to \mathbb{R}$ of length $\leq \omega$ sequences of reals by reals, in such a way that if x codes a sequence, then the length $\mathrm{lh}(x)$ of the sequence coded is uniquely determined, $(x)_i$ denotes the i-th element of the sequence, and x is the unique real so that $\langle (x)_i \rangle_{i<\mathrm{lh}(x)} = x$. By a tree on $\mathbb{R}$, we mean a functional $T^2 : \mathbb{R} \to 2$ so that T takes value 1 only on codes for finite sequences, so that $\{\langle x_0, \dots, x_i \rangle \mid T(\langle x_0, \dots, x_i \rangle) = 1\}$ is a tree in the usual sense.

Definition 3.1. Let $\mathcal{M} = (M_0, M_1, M_2)$ be an L^3 structure. We say $\mathcal{M}$ is a *β-model* if $M_0 = \omega$, $M_1 \subseteq \omega^\omega = \mathbb{R}$, and $M_2 \subseteq \omega^{M_1}$; and whenever T^2 is a (functional in M_2 coding) a tree on M_1, if T has an infinite branch, then $\mathcal{M}$ satisfies $(\exists x^1)(\forall k)(x)_k \subsetneq (x)_{k+1} \wedge T((x)_k) = 1$.

That is, trees on $\mathbb{R}^{\mathcal{M}}$ in $\mathcal{M}$ are wellfounded (in V) if and only if they are wellfounded in $\mathcal{M}$. We would have liked to define $\mathcal{M}$ to be a β-model if for any Σ^1_1 formula $\exists x^1 \phi(x, y, F)$ with parameters from $M_1 \cup M_2$, we have, for any $y \in M_1$, $F \in M_2$, that $\mathcal{M} \models \exists x \phi(x, y, F)$ if and only if $\exists x \phi(x, y, F)$ is true; but we must be careful about what we mean by "true". For, if x is a real not in M_1, then the value $F(x)$ is not defined. There are a number of ways to get around this, e.g., by appropriately altering the language L^3 and our base theory to accommodate a built-in coding of sequences of reals by reals. But it is more straightforward in our case to use the definition of β-model above.

We will be primarily interested in models of fragments of set theory, considered as β-models of third-order arithmetic. If $\mathcal{M} = (M, \in)$ is a transitive set with $\omega \in M$, we will refer to $\mathcal{M}$ as a model of third-order arithmetic, keeping in mind we are really referring to the structure $(\omega, M \cap \omega^\omega, M \cap \omega^{M \cap \mathbb{R}})$. It is immediate from our definitions that L_θ is a β-model. Indeed, whenever α is an ordinal with $\omega_1^{L_\theta} < \alpha \leq \theta$, then L_α is a β-model; this follows from the fact that branches through trees on $\mathbb{R}$ are themselves (coded by) reals. Consequently, taking collapses of Skolem hulls, we have many β-models L_γ with $\gamma < \omega_1^{L_\theta}$.

Our aim is to show that L_θ can be recovered from certain β-models, from which it will follow that L_θ is the minimal β-model of $\Delta^{\mathbb{R}}_1$-DET. Our starting point is a connection between $\Delta^{\mathbb{R}}_1$-DET and the third-order analogue of ATR_0.

DEFINITION 3.2. Π^1_∞-$\mathsf{TR}_\mathbb{R}$ is the theory in third-order arithmetic that asserts the following, for every Π^1_n formula $\phi(x^1, Y^2)$ with the displayed free variables. Suppose $W \subseteq \mathbb{R}\times\mathbb{R}$ is a regular relation. Then there is a functional $\theta : \mathbb{R}\times\mathbb{R} \to 3$ so that

$$(\forall a^1 \in \mathrm{dom}(W))(\forall x^1)\theta(a,x) = \begin{cases} 1 \text{ if } \phi(x, \theta \restriction \{b \mid \langle b, a\rangle \in W\}), \\ 0 \text{ otherwise.} \end{cases}$$

Here for $A \subseteq \mathbb{R}$, $\theta \restriction A$ denotes the functional θ' so that for all x, if $b \in A$, $\theta'(b,x) = \theta(b,x)$, and if $b \notin A$ then $\theta'(b,x) = 2$.

Note here we regard a functional $W : \mathbb{R} \to \omega$ as a binary relation if it determines the characteristic function of one; i.e., if there is a set $\mathrm{dom}(W) \subseteq \mathbb{R}$ so that $W(x) < 2$ whenever $x = \langle a, b\rangle$ for some $a, b \in \mathrm{dom}(W)$, and otherwise $W(x) = 2$. A binary relation is *regular* if whenever $A \subseteq \mathrm{dom}(W)$ is non-empty, there is some W-minimal $a \in A$. Be warned: we will routinely conflate the functionals of third-order arithmetic and the subsets of $\mathbb{R}, \mathbb{R}^{<\omega}, \mathbb{R}^\omega$, etc., which these functionals represent.

The idea of Π^1_∞-$\mathsf{TR}_\mathbb{R}$ is that for each $a \in \mathrm{dom}(W)$, the map $x \mapsto \theta(a,x)$ is the characteristic function of the set of reals obtained by iterating the defining formula ϕ along the wellfounded relation W on $\mathbb{R}$ up to a. Note that strictly speaking, Π^1_∞-$\mathsf{TR}_\mathbb{R}$ is projective *wellfounded recursion*, in that the relation W along which we iterate is not required to be a wellorder. This suits our purposes because we will iterate definitions along wellfounded trees on $\mathbb{R}$; taking the Kleene-Brouwer ordering of such a tree requires a wellordering of $\mathbb{R}$, but we would like to use as little choice as possible.

The following lemma makes reference to $\mathsf{TR}_1(\mathbb{R})$, introduced also in [7]. This is the restriction of Π^1_∞-$\mathsf{TR}_\mathbb{R}$ to the case that ϕ is Σ^1_1 and W is a wellorder.

LEMMA 3.3. *The following theories are equivalent over* RCA^3_0*:*

(1) $\Delta^\mathbb{R}_1$-DET*;*
(2) $\mathsf{TR}_1(\mathbb{R}) + \mathsf{SF}(\mathbb{R})$*;*
(3) Π^1_∞-$\mathsf{TR}_\mathbb{R} + \mathsf{SF}(\mathbb{R})$.

PROOF. Clearly, (3) implies (2). The equivalence of (1) and (2) is proved in [7]; and the proof that (1) implies the Σ^1_1 case in (3) is the essentially same proof given there for $\mathsf{TR}_1(\mathbb{R})$ with the appropriate adjustments. So all that is left to show is that Σ^1_1-wellfounded recursion implies Π^1_∞-$\mathsf{TR}_\mathbb{R}$.

So suppose inductively that we have Σ^1_n-wellfounded recursion, that W is a wellfounded relation on $\mathbb{R}$, and that $\phi(w^1, x^1, Y^2)$ is a Π^1_n formula. We wish to prove the instance of wellfounded recursion along W with formula $(\exists w)\phi$.

We define $\bar{W}$ to be a binary relation on $\omega \times \mathbb{R}$ so that $\bar{W}$ is isomorphic to the product $3 \times W$; namely,

$$\bar{W}(i^\frown x, j^\frown y) = \begin{cases} 1 & \text{if } i, j < 3 \text{ and } W(x,y) = 1 \text{ or } x = y \text{ and } i < j \\ 0 & \text{if } i, j < 3 \text{ and } W(x,y) \neq 1 \text{ and } x \neq y \text{ or } i \geq j \\ 2 & \text{in all other cases.} \end{cases}$$

The idea is to iterate Σ^1_n-wellfounded recursion along $\bar{W}$, breaking up into the three stages of applying $\neg\phi$, taking complements, and taking projections. So let us define the formula $\bar{\phi}(z, Y)$ by

$$\begin{aligned}\bar{\phi}(z,Y) \iff & (\exists i^0, a^1) a \in \operatorname{dom}(W), Y(i^\frown a, x) = 2 \text{ and} \\ & i = 0, (\exists w^1, x^1) z = \langle w, x\rangle, \text{ and } \neg\phi(w, x, [\langle b, y\rangle \mapsto Y(2^\frown b, y)]); \text{ or} \\ & i = 1, (\exists w^1, x^1) z = \langle w, x\rangle, \text{ and } Y(0^\frown a, z) = 0; \text{ or} \\ & i = 2 \text{ and } (\exists w^1) Y(1^\frown a, \langle w, z\rangle) = 1.\end{aligned}$$

To see $\bar{\phi}$ is Σ^1_n, it is enough to show the relation $\neg\phi(w, x, [\langle b, y\rangle \mapsto Y(2^\frown b, y)])$ is Σ^1_n (as a relation on w, x, Y). But this follows from the fact (checked to be provable in RCA^3_0) that if Y' is a functional Π^0_∞-definable from Y, then for any Σ^1_n formula π, there is, uniformly in π and the definition of Y' from Y, a Σ^1_n formula π', so that

$$(\forall x^1)\pi'(x, Y) \iff \pi(x, Y').$$

We obtain the result by applying Σ^1_n-wellfounded recursion to $\bar{W}$ with $\bar{\phi}$. From the θ obtained, the desired instance of Σ^1_{n+1} recursion is witnessed by the relation $\langle a, x\rangle \mapsto \theta(2^\frown a, x)$ (which exists by Δ^0_1-Comprehension). $\dashv$

We remark that the uniqueness of the functional θ is provable from the Σ^1_1-Comprehension scheme (which itself follows from $\mathsf{TR}_1(\mathbb{R})$), using regularity of the relation W applied to $\{a \in \operatorname{dom}(W) \mid (\exists x^1)\theta_1(a, x) \neq \theta_2(a, x)\}$.

§4. From β-models to set models. In this section we show that from any β-model $\mathcal{M}$ of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, one can define a transitive set model M^{set} with the same reals and functionals; and furthermore, any set model so obtained contains L_θ as a subset. By what we have shown, L_θ is a β-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, so this proves that L_θ is the minimal β-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$.

These results are essentially a recapitulation in the third-order context of the correspondence between β-models of ATR_0 and wellfounded models $\mathsf{ATR}^{\text{set}}_0$ described in Chapter VII.3-4 of [8]; therefore we omit most details, taking care mainly where the special circumstances of the third-order situation arise.

Let $\mathcal{M}$ be a L^3-structure modelling Π^1_∞-$\mathsf{TR}_\mathbb{R}$. Working inside $\mathcal{M}$, we say $T : \mathbb{R} \to \omega$ is a *suitable tree* if

1. T codes a tree on $\mathbb{R}$,
2. T is non-empty, i.e. $T(\langle\rangle) = 1$, and
3. T is regular: if $A \subseteq T$, there is $a \in A$ with no proper extension in A.

The third item is understood to quantify over type-2 objects corresponding to characteristic functions of subsets of T. We take suitable trees to be regular because this is what's required by Π^1_∞-$\mathsf{TR}_\mathbb{R}$ and is possibly stronger than non-existence of a branch; of course the two are equivalent assuming $\mathsf{DC}_\mathbb{R}$, in particular, in β-models.

Now suppose $\mathcal{M}$ is a β-model. If T is a tree on $\mathbb{R}^\mathcal{M}$ coded by some functional in M_2, then T is suitable in $\mathcal{M}$ if and only if T is (non-empty and) wellfounded. We will define M^{set} to be the set of collapses of suitable trees in $\mathcal{M}$. Namely, given a wellfounded tree, define by recursion on the wellfounded relation $\supsetneq \cap (T \times T)$,

$$f(s) = \{f(s^\frown\langle a\rangle) \mid a \in \mathbb{R} \wedge s^\frown\langle a\rangle \in T\}.$$

Then put $|T| = f(\langle\rangle)$. Notice that $|T|$ need not be transitive, as we only take $f(s)$ to be the pointwise image of *one-step* extensions of s. We define

$$M^{\text{set}} = \{|T| \mid T \in M_2 \text{ is a suitable tree}\}.$$

Such M^{set} is transitive: If T is a suitable tree in $\mathcal{M}$ then any $x \in |T|$ is $|T_s|$ for some $s \in T$. But $T_s = \{t \mid s^\frown t \in T\}$ is evidently a suitable tree, and belongs to $\mathcal{M}$ by Δ^0_1-Comprehension.

Although we are interested primarily in β-models of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, it is worth making a definition of M^{set} that works for ω-models of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, that is, models $\mathcal{M}$ with standard ω so that $M_1 \subseteq \mathbb{R}$ and $M_2 \subseteq \omega^{M_1}$. Working inside such an $\mathcal{M}$, say that $\mathrm{ISO}(T^2, X^2)$ holds, where T is a suitable tree, if $X \subseteq T \times T$ and for all $s, t \in T$, we have

$$\begin{aligned}\langle s,t\rangle \in X \iff (\forall x^1)[&s^\frown\langle x\rangle \in T \to (\exists y^1)(t^\frown\langle y\rangle \in T \wedge \langle s^\frown\langle x\rangle, t^\frown\langle y\rangle\rangle \in X)\\ &\wedge t^\frown\langle x\rangle \in T \to (\exists y^1)(s^\frown\langle y\rangle \in T \wedge \langle s^\frown\langle y\rangle, t^\frown\langle x\rangle\rangle \in X)].\end{aligned}$$

(The point is, $\langle s,t\rangle \in X$ if and only if $|T_s| = |T_t|$). The existence and uniqueness of an X so that $\mathrm{ISO}(T, X)$ holds is provable in Π^1_∞-$\mathsf{TR}_\mathbb{R}$, using the fact that T is suitable. Letting $\bar{n}$ denote the real $\langle n, n, n, \ldots\rangle$, we may define $S \oplus T$, for suitable trees S, T, as the set of sequences $\{\langle\bar{0}\rangle^\frown s \mid s \in S\} \cup \{\langle\bar{1}\rangle^\frown t \mid t \in T\}$. Then set $S =^* T$ iff for the unique X with $\mathrm{ISO}(S \oplus T, X)$, we have $\langle\langle\bar{0}\rangle, \langle\bar{1}\rangle\rangle \in X$; and set $S \,\epsilon\, T$ iff for the unique X with $\mathrm{ISO}(S \oplus T, X)$, there is some real x so that $\langle\langle\bar{0}\rangle, \langle\bar{1}, x\rangle\rangle \in X$. Then provably in Π^1_∞-$\mathsf{TR}_\mathbb{R}$, $=^*$ is an equivalence relation on the class of suitable trees, and ϵ is well-defined and extensional relation on the equivalence classes $[T]_{=^*}$, so inducing a relation $\in^*$ on these. We define

$$M^{\text{set}} = \langle\{[T]_{=^*} \mid T \in M_2 \text{ is a suitable tree in } \mathcal{M}\}, \in^*\rangle.$$

For β-models $\mathcal{M}$, the M^{set} we obtain is a wellfounded structure, and is isomorphic to the transitive set M^{set} defined above, via the map $[T]_{=^*} \mapsto |T|$. For brevity, we will from now on refer to $[T]_{=^*}$ as $|T|$ (even for T in non β-models, so that T may be illfounded in V).

Recall now some basic axiom systems in the language of set theory. BST is the theory consisting of Extensionality, Foundation, Pair, Union, and Δ_0-Comprehension. Axiom Beta, which we denote $\mathrm{Ax}\,\beta$, states that every regular relation r has a collapse map; that is, a map $f : \mathrm{dom}(r) \to V$ so that for all $x \in \mathrm{dom}(r)$, $f(x) = \{f(y) \mid \langle y, x\rangle \in r\}$.

PROPOSITION 4.1. *Let $\mathcal{M}$ be an ω-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$. Then*

1. *M^{set} is an ω-model of $\mathsf{BST} + \mathrm{Ax}\,\beta +$ "$\mathcal{P}(\omega)$ exists".*
2. *$\mathcal{M}$ and M^{set} have the same reals $x : \omega \to \omega$ and functionals $F : \mathbb{R} \to \omega$; that is, $M_1 = \mathbb{R} \cap M^{\text{set}}$ and $M_2 = (\omega^{\mathbb{R}\cap M^{\text{set}}}) \cap M^{\text{set}}$.*
3. *In M^{set}, every set is hereditarily of size at most 2^ω; that is, for all $x \in M^{\text{set}}$, there is an onto map $f : \mathcal{P}(\omega)^{M^{\text{set}}} \to \mathrm{tcl}(x)$ in M^{set}, where $\mathrm{tcl}(x)$ denotes the transitive closure of x.*
4. *If $\alpha \in \mathrm{ON}^{M^{\text{set}}}$, then $M^{\text{set}} \models$ "L_α exists"; furthermore, $L_\alpha^{M^{\text{set}}} = L_\alpha$ when α is in the wellfounded part of M^{set}.*
5. *M^{set} is wellfounded if and only if $\mathcal{M}$ is a β-model.*

PROOF. (1) Since $\mathcal{M}$ is an ω-model of RCA_0^3, the tree

$$\{\langle \bar{n_0}, \bar{n_1}, \ldots, \bar{n_k}\rangle \mid (\forall i < k) n_{i+1} < n_i\}$$

belongs to M_2. Clearly it is a suitable tree in $\mathcal{M}$, and $|T| \in M^{\mathrm{set}}$ is the ω of M^{set}. That $\mathcal{P}(\omega)$ exists in M^{set} is a similar exercise in coding: given any real x, there is a canonical tree $T(x)$ so that $|T(x)| = x$, membership in $T(x)$ being uniformly Π^1_∞-definable from x; and from any suitable tree collapsing to a real, one can define in Π^1_∞-$\mathsf{TR}_\mathbb{R}$ the x it collapses to. So $\mathcal{P}(\omega)^{M^{\mathrm{set}}}$ is precisely $|T|$, where $T = \{\langle\rangle\} \cup \{\langle x\rangle^\frown s \mid s \in T(x)\}$.

For the axioms of BST, Extensionality follows from the fact that the relation $\in^*$ is extensional on M^{set}. Pair and Union are straightforward, only requiring Σ^1_1-Comprehension to show that from given suitable trees $S, T \in \mathcal{M}_2$, one can define trees corresponding to $\{|S|, |T|\}$ and $\bigcup |S|$.

Δ_0-Comprehension is similar. Notice here that although the relations $=*$ and $\in^*$ are in general Σ^2_1, when restricted to a given tree T with parameter X witnessing $\mathrm{ISO}(T, X)$, the relations $|T_s| =^* |T_t|$ and $|T_s| \in^* |T_t|$, regarded as binary relations T, are each Π^1_2 in the parameters T, X. From this, one shows by induction on formula complexity that for any Δ_0 formula $\phi(u_1, \ldots, u_k)$ in the language of set theory, the k-ary relation on T defined by

$$P(s_1, \ldots, s_k) \iff M^{\mathrm{set}} \models \phi(|T_{s_1}|, \ldots, |T_{s_k}|)$$

is Π^1_n for some n (again, in the parameter X). Δ_0-Comprehension is then straightforward to prove.

For Foundation, suppose towards a contradiction T is a suitable tree so that in M^{set}, $|T|$ is a non-empty set with no $\in^*$-minimal element. Let X witness $\mathrm{ISO}(T, X)$. Then

$$A = \{s \in T \mid (\exists x^1)\langle x\rangle \in T \wedge \langle\langle x\rangle, s\rangle \in X\}$$

is a set of nodes in T such that every element of A can be properly extended in A. This contradicts suitability of T.

$\mathrm{Ax}\,\beta$ is in a similar vein. Given a suitable tree R so that $|R| = r$ is a regular relation in M^{set}, verify that the relation $W = \{\langle s, t\rangle \in R \times R \mid \langle |R_s|, |R_t|\rangle \in r\}$ is a regular relation in $\mathcal{M}$. A tree F so that $|F| : \mathrm{dom}(r) \to \mathrm{ON}$ is precisely the collapse map is then defined by Π^1_∞-$\mathsf{TR}_\mathbb{R}$ along the relation W.

(2) The inclusion $\subseteq$ is another coding exercise. The reverse follows from Δ_0-Comprehension in M^{set}.

(3) Define a suitable F so that $f = |F| \supseteq \{\langle s, |T_s|\rangle \mid s \in T\}$.

(4) The construction is very nearly identical to that of Lemma VII.4.2 of [8]. The only modifications are that we work in Π^1_∞-$\mathsf{TR}_\mathbb{R}$, and so do not induct along a wellorder; rather, we induct along the suitable tree A for which $\alpha = |A|$. The ramified language we define therefore makes use of variables v_i^a, where $i \in \omega$ and $a \in A$, intended to range over $L^{M^{\mathrm{set}}}_{|T_a|}$. The rest of the proof is unchanged.

(5) Evidently if $\mathcal{M}$ is a β-model, every suitable tree in $\mathcal{M}$ is in fact wellfounded, so that $\in^*$ is a wellfounded relation. Conversely, if $\mathcal{M}$ is not a β-model, there some tree T which $\mathcal{M}$ thinks is suitable, but is not wellfounded. Then if $\langle s_n\rangle_{n\in\omega}$ is a branch through T, the sequence $\langle |T_{s_n}|\rangle_{n\in\omega}$ witnesses illfoundedness of $\in^*$. $\dashv$

THEOREM 4.2. *Let $\mathcal{M}$ be a β-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$. Then $L_\theta \subseteq M^{\mathrm{set}}$.*

PROOF. Work in M^{set}. Notice that ω_1 exists by an application of $\text{Ax}\,\beta$ to the regular relation $\{\langle x, y\rangle \mid x, y$ are wellorders of ω with x isomorphic to an initial segment of $y\}$. Now if $\omega_1^L < \omega_1$, we're done, since L_{ω_1} is then a model of ZF + "$\mathcal{P}(\omega)$ exists", so θ must exist and be less than ω_1. So we can suppose $\omega_1^L = \omega_1$.

We have that every tree on $\mathcal{P}(\omega)$ is either ranked or illfounded; we claim the same is true in L. For suppose $T \in L$ is a tree on $\mathcal{P}(\omega) \cap L$. If T is ranked, then let $\rho : T \to \text{ON}$ be the ranking function. Let α be large enough that $T \in L_\alpha$. Then it is easily checked that $\rho \in L_{\alpha+\omega\cdot\rho(\emptyset)}$; note the latter exists because (by $\text{Ax}\,\beta$) the ordinals are closed under ordinal $+$ and $\cdot$.

Now suppose T is illfounded. Then let $x = \langle x_i\rangle_{i\in\omega}$ be a branch through T. Note that each $x_i \in L$, hence in L_{ω_1}. Let α_i be sufficiently large that $x_i \in L_{\alpha_i}$. Since $\omega_1 = \omega_1^L$, the map $i \mapsto \alpha_i$ is bounded in ω_1^L (note M^{set} models $\mathsf{DC}_{\mathbb{R}}$, so ω_1 is regular in M^{set}). So we have some admissible level L_γ with $\gamma < \omega_1$ so that $\alpha = \sup_{i\in\omega} \alpha_i < \gamma$; but then $T \cap L_\alpha$ is an illfounded tree, so has some branch definable over L_γ. So we have a branch through T in L. $\dashv$

§5. Higher levels. For a transitive set U, let $\Delta_1(U)$-DET and $\Sigma_1(U)$-DET denote, respectively, clopen and open determinacy for game trees $T \subseteq U^{<\omega}$. We recall from [2] the principles Π_1-$\mathsf{RAP}(U)$:

DEFINITION 5.1. Let U be a transitive set. The Π_1-*Reflection to Admissibles Principle for* U (denoted Π_1-$\mathsf{RAP}(U)$) is the assertion that $\mathcal{P}(U)$ exists, together with the following axiom scheme, for all Π_1 formulae $\phi(u)$ in the language of set theory: Suppose $Q \subseteq \mathcal{P}(U)$ is a set and $\phi(Q)$ holds. Then there is an admissible set M so that

- $U \in M$.
- $\bar{Q} = Q \cap M \in M$.
- $M \models \phi(\bar{Q})$.

For $n \in \omega$, let θ_n be the least ordinal so that L_{θ_n} is a model of "$\mathcal{P}^n(\omega)$ exists" plus Π_1-$\mathsf{RAP}(\mathcal{P}^n(\omega))$; note $\theta = \theta_0$, and by the definition of Π_1-$\mathsf{RAP}(U)$, $L_{\theta_n} \models$ "$\mathcal{P}^{n+1}(\omega)$ exists" + "ω_{n+1} is the largest cardinal". Furthermore, L_{θ_n} Σ_1-projects to ω with parameter $\{\omega_{n+1}\}$, and we have the following characterisation of the ordinals θ_n in terms of trees:

PROPOSITION 5.2. *Say T is a tree on $\mathcal{P}^{n+1}(\omega), \mathcal{P}^n(\omega)$ if whenever $s \in T$, we have $s_{2n} \in \mathcal{P}^{n+1}(\omega)$ and $s_{2n+1} \in \mathcal{P}^n(\omega)$. Consider a closed game on such a tree, that is, a game where players cooperate to choose a branch through the tree, and player I wins precisely the infinite plays. Then θ_n is the least ordinal so that L_θ satisfies "for every tree T on $\mathcal{P}^{n+1}(\omega), \mathcal{P}^n(\omega)$, either I wins the closed game on T, or the game is ranked for player II".*

Note that a winning strategy for I in such a game is (coded by) an element of $\mathcal{P}^{n+1}(\omega)$; a ranking function for II (the open player) is a partial function $\rho : T \rightharpoonup \text{ON}$ so that $\rho(\emptyset)$ exists, and whenever $s \in T$ has even length and $\rho(s)$ is defined, we have $(\forall x)(\exists y) s^\frown\langle x\rangle \in T \to \rho(s^\frown\langle x, y\rangle) < \rho(s)$.

We obtain a generalization of Schweber's separation result to higher types by looking at the models L_{θ_n}:

THEOREM 5.3. *For* $n \in \omega$, L_{θ_n} *is a model of* $\Delta_1(\mathcal{P}^{n+1}(\omega))$-DET, *but not of* $\Sigma_1(\mathcal{P}^{n+1}(\omega))$-DET.

PROOF. The proof of $\Delta_1(\mathcal{P}^{n+1}(\omega))$-DET is exactly like that of $\Delta^{\mathbb{R}}_1$-DET in Theorem 2.1: given a parameter set Q coding a wellfounded tree T on $\mathcal{P}^{n+1}(\omega)$, if neither player wins the game on T, reflect this Π_1 statement to an admissible set M containing $\mathcal{P}^n(\omega)$. Use the fact that $T \cap M \in M$ is wellfounded to contradict admissibility.

To see that $\Sigma_1(\mathcal{P}^{n+1}(\omega))$-DET fails, again define a game where the open player proposes a Σ_1 formula $\phi(\omega_{n+1})$, and the closed player chooses a truth value and plays approximations to the model L_{θ_α} (now using the characterization of Proposition 5.2, closing under the operation sending a game tree on $\mathcal{P}^{n+1}(\omega), \mathcal{P}^n(\omega)$ to a winning strategy for I or ranking function for II, whichever exists), while player II lists elements of $\mathcal{P}^{n+1}(\omega)$ that must be included in the model. As before II has no winning strategy in V, so none in L_{θ_n}, and any winning strategy for I computes the $\Sigma_1(\{\omega_{n+1}\})$ theory of L_{θ_n}, so cannot belong to L_{θ_n}. ⊣

Note that we haven't attempted to give these results in the context of some standard base theory of n-th order arithmetic, but the models L_{θ_n}, being models of BST, should clearly be models of any reasonable such base theory.

§6. Conclusions. We have shown that $\Sigma^{\mathbb{R}}_1$-DET, Σ^0_4-DET, and $\Delta^{\mathbb{R}}_1$-DET are strictly decreasing in consistency strength when we require the models under consideration to satisfy some mild absoluteness. For by the results of the last section, any β-model of $\Sigma^{\mathbb{R}}_1$-DET contains a copy of L_θ, and the argument of Theorem 2.1 then applies; it follows that any β-model of $\Sigma^{\mathbb{R}}_1$-DET must contain the Σ_1-theory of L_θ, from which winning strategies in Σ^0_4 games are computable. So a β-model of $\Sigma^{\mathbb{R}}_1$-DET always satisfies Σ^0_4-DET, in fact, (boldface) $\boldsymbol{\Sigma}^0_4$-DET.

Now, we have (working in Π^1_1-CA_0) that Σ^0_4-DET is equivalent to the existence of a β-model of $\Delta^{\mathbb{R}}_1$-DET, so Σ^0_4-DET is (consistency strength-wise) strictly stronger than $\Delta^{\mathbb{R}}_1$-DET. But it is unclear whether $\Sigma^{\mathbb{R}}_1$-DET outright implies the existence of a β-model of $\Delta^{\mathbb{R}}_1$-DET, that is, whether Σ^0_4-DET is provable from the third order theory $\Sigma^{\mathbb{R}}_1$-DET.

Indeed, Schweber asks (Question 5.1 of [7]) whether $\Sigma^{\mathbb{R}}_1$-DET and $\Delta^{\mathbb{R}}_1$-DET have the same second order consequences, and Σ^0_4-DET would be an interesting counterexample. However, the present study doesn't rule out the possibility that there are (necessarily non-β-) models of $\Sigma^{\mathbb{R}}_1$-DET in which Σ^0_4-DET fails. One can show that there is no model of $\Sigma^{\mathbb{R}}_1$-DET whose reals are precisely those of L_θ, and so any such (set) model will be illfounded with wellfounded part well below θ. The problem of constructing such a model (if one exists) then seems a difficult one.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90024, USA
E-mail: shac@math.ucla.edu