Problem 1. Section 2.2/ Exercise 8.
Suppose Σ is a set of formulas such that for all sentences τ, either Σ |− τ or Σ |− ¬τ. Assume that A |− Σ. Show that for any τ, A |− τ iff Σ |− τ.

For one of the directions, suppose that Σ |− τ. Then by definition of logical implication and since A |− Σ, it follows that A |− τ.

For the other direction, suppose that A |− τ. Then A |− ¬τ. By the definition of logical implication and since A |− Σ, it follows that Σ |− ¬τ. So, by the assumptions on Σ, we get that Σ |− τ.

Problem 2. Section 2.2/ Exercise 9.
Assume that the language has equality and a two-place predicate P. For each of the following find a sentence σ such that the model A satisfies σ iff the condition is met.

(1) The universe of A has exactly two members.
σ = "∃x∃y(¬(x = y) ∧ ∀z(z = x ∨ z = y))"

(2) P is a function (from the universe of A to itself).
σ = "∀x∀y(P(x, y) ∧ ∀z(P(x, z) → y = z))"

(3) P is a permutation of the universe of A (i.e. it is a one-to-one, onto function).
σ f = "∀x∀y(P(x, y) ∧ ∀z(P(x, z) ∧ P(y, z)) → x = y)" and σ 11 = "∀x∀y∃z(P(x, z) ∧ P(y, z)) → x = y)" and σ o = "∀y∃xP(x, y)". Then σ f says that P is a function, σ 11 says that it is one-to-one, and σ o says that it is onto. Set σ = σ f ∧ σ 11 ∧ σ o.

Problem 3. Section 2.2/ Exercise 11.
Give a formula to define each of the following in (N; +, ·).

Part (a): {0}.
φ = "x + x = x"

Part (b): {1}.
φ = "x · x = x ∧ (x + x ≠ x)"

Part (c): {⟨m, n⟩ | n is the successor of m in N}.
φ(x, y) = "∃z(z · z = z ∧ z + z ≠ z ∧ x + z = y)"

Part (d): {⟨m, n⟩ | m < n in N}.
φ(x, y) = "∃z(z + z ≠ z ∧ x + z = y)"

Problem 4. Section 2.4/ Exercise 2.
To which axiom groups, if any, do each of the following formulas belong?
**Part (a):** \([\forall x P x \rightarrow \forall y P y \rightarrow P z] \rightarrow [\forall x P x \rightarrow (\forall y P y \rightarrow P z)]\) is a tautology, and so it is in group 1 of the logical axioms.

**Part (b):** \(\forall y [\forall x (P x \rightarrow P x) \rightarrow (P c \rightarrow P c)]\) is an axiom from group 2. (Since \(c\) is substitutable for \(x\) in \((P x \rightarrow P x))\).

**Part (c):** \(\forall x \exists y P x y \rightarrow \exists y P y y\). This is not an axiom, since \(y\) is not substitutable for \(x\) in \(\exists y P x y\).

**Problem 5.** Section 2.4/ Exercise 12.

**Part (a):** Show that \(\vdash \alpha \rightarrow \beta\), then \(\Delta \vdash \alpha \rightarrow \forall \beta\). Namely, a deduction to \(k = 0\), then \(\Delta \vdash \alpha \rightarrow \forall \beta\).

**Part (b):** Show that it is not in general true that \(\alpha \rightarrow \beta \vdash \forall \alpha \rightarrow \forall \beta\).

**Solutions:**

**Part (a):** Suppose that \(\vdash \alpha \rightarrow \beta\). By the generalization theorem, it follows that \(\forall \alpha \rightarrow \forall \beta\). Also, the following is a deduction from \(\forall \alpha \rightarrow \forall \beta\) to \(\forall \alpha \rightarrow \forall \beta\):

- \(\forall \alpha (\alpha \rightarrow \beta)\)
- \(\forall \alpha (\alpha \rightarrow \beta) \rightarrow (\forall \alpha \rightarrow \forall \beta)\)
- \((\forall \alpha \rightarrow \forall \beta)\).

When we add the above to the deduction to \(\forall \alpha \rightarrow \forall \beta\), we obtain a deduction to \(\forall \alpha \rightarrow \forall \beta\).

**Part (b):** Let \(\mathfrak{A} = (\mathbb{N}, <)\). Let \(\alpha = \exists y (x < y)\), \(\beta = \forall y (x = y \lor x < y)\). Let \(s\) be a variable assignment such that \(s(x) = 0\). Then \(\mathfrak{A} \models \beta[s]\), and so \(\mathfrak{A} \models (\alpha \rightarrow \beta)[s]\). Also \(\mathfrak{A} \models \forall \alpha [s]\), but \(\mathfrak{A} \not\models \forall \beta[s]\). So, \(\mathfrak{A} \not\models \forall \alpha \rightarrow \forall \beta[s]\).

The crucial difference between the assumptions in part (a) and (b) is that in part (a) we don’t consider a truth assignment. Namely, a deduction to \(\alpha \rightarrow \beta\) does not depend on \(s\) and so \(\vdash \alpha \rightarrow \beta\) is a much stronger assumption than \(\mathfrak{A} \models (\alpha \rightarrow \beta)[s]\). Also, note that \(\alpha \rightarrow \beta \not\models \forall \alpha \rightarrow \forall \beta\).

**Problem 6.** Section 2.4/ Exercise 12.

Suppose \(\Gamma\) is a consistent set of formulas. Show that \(\Gamma\) can be extended to a consistent set \(\Delta\) such that for any formula \(\alpha\), either \(\alpha \in \Delta\) or \(\neg \alpha \in \Delta\).

**Solution:**

Enumerate all formulas: \(\{\alpha_k \mid k \geq 0\}\). Recursively define a sequence \(\{\Delta_k \mid k \geq 0\}\) as follows:

- \(\Delta_0 = \Gamma\).
- \(\Delta_{k+1} = \Delta_k \cup \{\alpha_k\}\) if \(\Delta_k \vdash \alpha_k\) and \(\Delta_{k+1} = \Delta_k \cup \{\neg \alpha_k\}\) otherwise.

Set \(\Delta = \bigcup_{k \geq 0} \Delta_k\). Then clearly \(\Gamma \subseteq \Delta\) and by definition, for each formula, \(\alpha\), either \(\alpha \in \Delta\) or \(\neg \alpha \in \Delta\). It remains to show that \(\Delta\) is consistent. To that effect we will show that each \(\Delta_k\) is consistent by induction on \(k\). If \(k = 0\), then \(\Delta_0 = \Gamma\) is consistent. Now suppose that \(\Delta_k\) is consistent; we have to show that \(\Delta_{k+1}\) is consistent. Suppose for contradiction that for some formula \(\beta\), \(\Delta_{k+1} \vdash \beta \land \neg \beta\). There are two cases:

**Case 1:** \(\Delta_{k+1} = \Delta_k \cup \{\alpha_k\}\). In that case \(\Delta_k \vdash \alpha_k\). Since \(\Delta_{k+1} \vdash \beta \land \neg \beta\), by the deduction theorem we have that \(\Delta_k \vdash \alpha_k \rightarrow (\beta \land \neg \beta)\). But \(\Delta_k \vdash \alpha_k\), so \(\Delta_k \vdash (\beta \land \neg \beta)\), which is a contradiction since \(\Delta_k\) is consistent.
Case 2: $\Delta_{k+1} = \Delta_k \cup \{\neg \alpha_k\}$. In that case $\Delta_k \not\vdash \alpha_k$. Since $\Delta_{k+1} \vdash \beta \land \neg \beta$, by the deduction theorem we have that $\Delta_k \vdash \neg \alpha_k \rightarrow (\beta \land \neg \beta)$. By rule T this means that $\Delta_k \vdash (\neg \beta \lor \beta) \rightarrow \alpha_k$. But $(\neg \beta \lor \beta)$ is a tautology, so (by rule T) we have that $\Delta_k \vdash \alpha_k$, contradiction with being in case 2.

Thus each $\Delta_k$ is consistent and since every deduction is finite, we have that $\Delta$ is consistent. (Here we use that if there is a deduction from $\Delta$ to some formula $\phi$, for some $k$ the deduction will be using only formulas from $\Delta_k$ and so there is a deduction from $\Delta_k$ to $\phi$.)