Problem 1. Section 2.5/ Exercise 2

Prove the equivalence of parts (a) and (b) of the completeness theorem.

Solution:

(a) ⇒ (b): Suppose (a) holds and Γ is consistent. We have to show that Γ is satisfiable. Suppose otherwise i.e. for any model A, A |≠ Γ. Then we have (vacuously) that Γ |= β ∧ ¬β. Then by (a) it follows that Γ |= β ∧ ¬β. But that is a contradiction, since Γ was assumed to be consistent. Therefore Γ is satisfiable.

(b) ⇒ (a): Suppose (b) holds and Γ |= φ. We have to show that Γ ⊢ φ. Suppose otherwise i.e. Γ |= φ. Then Γ ∪ {¬φ} is consistent by RAA (Cor 24 E on pg 119). Then by (b) it follows that Γ ∪ {¬φ} is satisfiable i.e. there is some model A such that A |= Γ and A |= ¬φ. Therefore Γ |= φ, which is a contradiction with the assumptions. It follows that Γ |= φ.

Problem 2. Section 2.5/ Exercise 6

Let Σ₁ and Σ₂ be such that nothing is a model of both of them. Show there is a sentence τ such that ModΣ₁ ⊂ Modτ and ModΣ₂ ⊂ Mod¬τ.

Solution:

We have to find a sentence τ such that if Σ₁ |= τ and Σ₂ |= ¬τ.

Since Σ₁ ∪ Σ₂ is unsatisfiable, we have by the compactness theorem, that there is some finite set ∆ ⊂ Σ₁ ∪ Σ₂ such that ∆ is not satisfiable. Let ∆₁ = ∆ ∩ Σ₁ and ∆₂ = ∆ ∩ Σ₂. Enumerate ∆₁ = {φ₁,...,φₖ} and ∆₂ = {ψ₁,...,ψₙ}. Set τ₁ = φ₁ ∧ φ₂... ∧ φₖ and τ₂ = ψ₁ ∧ ψ₂... ∧ ψₙ. Define τ = τ₁ ∧ ¬τ₂.

Claim 1. Σ₁ |= τ

Proof. Suppose A |= Σ₁. Then A |= Δ₁, and so we have that A |= τ₁. Also since Δ is not satisfiable, we know that A |≠ Δ. Since A |= Δ₁, we have that A |≠ Δ₂ i.e. A |= ¬τ₂. Thus A |= τ.

Claim 2. Σ₂ |= ¬τ

Proof. Suppose A |= Σ₂. Then A |= Δ₂, and so A |= τ₂. Since ¬τ = ¬τ₁ ∨ τ₂, we get that A |= ¬τ.

Since Σ₁ |= τ, it follows that ModΣ₁ ⊂ Modτ. And since Σ₂ |= ¬τ, it follows that ModΣ₂ ⊂ Mod¬τ.
Problem 3. Section 2.5/ Exercise 7.
For each of the following show that there is a deduction or give a counter model:
(a): $\forall x (Qx \to \forall y Qy)$ 
(b): $\exists x P x \to \forall z (Pz \to Qz)$ 
(c): $\forall z (Pz \to Qz) \to (\exists x P x \to \forall y Qy)$ 
(d): $\neg \exists y \forall x (P xy \leftrightarrow \neg P xx)$

Solution:
(a): The following is a counter model. Let $A$ have as universe the natural numbers and for any natural number $n$, set $Q^A(n)$ iff $n$ is even. Then $A \not\models Q2 \to \forall y Qy$, and so $A \not\models \forall x (Qx \to \forall y Qy)$.
(b): We will show that a deduction exists. First, the following is a deduction from $\{ (\exists x P x \to \forall y Qy), Pz \}$ to $Qz$:
- $\forall x \neg P x \to \neg P z$ (logical axiom)
- $(\forall x \neg P x \to \neg P z) \to (P z \to \exists x P x)$ (tautology)
- $P z$ (given)
- $\exists x P x$ (MP)
- $(\exists x P x \to \forall y Qy)$ (given)
- $\forall y Qy$ (MP)
- $\forall y Qy \to Q z$ (logical axiom)
- $Q z$ (MP)
Since $\{ (\exists x P x \to \forall y Qy), Pz \} \vdash Qz$, by the deduction theorem it follows that $\vdash (\exists x P x \to \forall y Qy) \to P z \to Qz$. Then by the generalization theorem we get $\vdash (\exists x P x \to \forall y Qy) \to \forall z (Pz \to Qz)$. So, $\vdash (\exists x P x \to \forall y Qy) \to \forall z (Pz \to Qz)$.
(c): $\forall z (Pz \to Qz) \to (\exists x P x \to \forall y Qy)$ has a counter model. Let $A$ have as universe the natural numbers and for any natural number $n$, set $P^A(n)$ iff $n$ is divisible by 4 and $Q^A(n)$ iff $n$ is even. Then $A \models \forall z (Pz \to Qz)$ and $A \models \exists x P x$, but $A \not\models \forall y Qy$. It follows that $A \not\models \forall z (Pz \to Qz) \to (\exists x P x \to \forall y Qy)$.
(d): There is a deduction to $\neg \exists y \forall x (P xy \leftrightarrow \neg P xx)$. It suffices to show that $\vdash \forall y \exists x (\neg (P xy \leftrightarrow \neg P xx))$. By the generalization theorem it is enough to show that $\vdash \exists x (\neg (P xy \leftrightarrow \neg P xx))$ The deduction is as follows:
- $\forall x (P xy \leftrightarrow \neg P xx) \to (P yy \leftrightarrow \neg P yy)$ (logical axiom)
- $[\forall x (P xy \leftrightarrow \neg P xx) \to (P yy \leftrightarrow \neg P yy)] \to [\neg (P yy \leftrightarrow \neg P yy) \to \exists x (\neg (P xy \leftrightarrow \neg P xx))]$ (tautology)
- $[\neg (P yy \leftrightarrow \neg P yy) \to \exists x (\neg (P xy \leftrightarrow \neg P xx))]$ (MP)
- $\neg (P yy \leftrightarrow \neg P yy)$ (tautology)
- $\exists x (\neg (P xy \leftrightarrow \neg P xx))$ (MP)

Problem 4. Section 2.6/ Exercise 1
Show that the following sentences are finitely valid, i.e. true in every finite structure.
(a) $\exists x \exists y \exists z [(P x f x \to P x x) \lor (P x y \land P y z \land \neg P x z)]$. 
(b) \( \exists x \forall y \exists z [(Qzx \rightarrow Qzy) \rightarrow (Qxy \rightarrow Qxx)] \).

Solution:

(a). Suppose \( \mathfrak{A} \) is a finite model. Suppose for contradiction that \( \mathfrak{A} \) does not satisfy the sentence. Then:

\begin{enumerate}
  \item \( \mathfrak{A} \models \forall x \forall y \forall z ((Pxy \land Pyz) \rightarrow Pxz) \),
  \item \( \mathfrak{A} \models \forall x (Pxfx \land \neg Pxx) \),
\end{enumerate}

It follows that \( P^\mathfrak{A} \) is transitive, irreflexive and for every \( d \in |\mathfrak{A}| \) and natural number \( n \), such that \( d = f^\mathfrak{A} n(d) = f^\mathfrak{A}(f^\mathfrak{A}(...(d)...)) \) \( n \) many times. But then by applying transitivity and since for every \( i < n \), \( \mathfrak{A} \models \forall x ( Pf^i(x), f^{i+1}(x)) \), it follows that \( \langle d, d \rangle \in P^\mathfrak{A} \). But that is a contradiction with irreflexivity.

(b). Suppose that \( \mathfrak{A} \models \forall x \exists y \forall z [(Qzx \rightarrow Qzy) \land Qxy \land \neg Qxx] \) i.e. the negation of the given sentence. We have to show that this model is infinite. First note that this implies that for all \( d \in \mathfrak{A} \), \( \langle d, d \rangle \not\in Q^\mathfrak{A} \). Construct a sequence \( \langle a_n \mid n \geq 0 \rangle \) of elements of \( \mathfrak{A} \) as follows. Let \( a_0 \) be any element. Suppose we have defined \( a_n \), we define \( a_{n+1} \) to witness that \( \mathfrak{A} \models \exists y \forall z [(Qzx \rightarrow Qzy) \land Qxy \land \neg Qxx][a_n] \) i.e. when we plug in \( a_n \) for \( x \).

That means that in \( \mathfrak{A} \) we have:

\begin{enumerate}
  \item \( \langle a_n, a_{n+1} \rangle \in Q^\mathfrak{A} \),
  \item for all \( d \in |\mathfrak{A}| \), if \( \langle d, a_n \rangle \in Q^\mathfrak{A} \), then \( \langle d, a_{n+1} \rangle \in Q^\mathfrak{A} \).
\end{enumerate}

Since the relations is always irreflexive, the first item implies that \( a_n \neq a_{n+1} \).

The second item implies that \( a_{n+1} \) must be different than all of the \( a_i \)'s for \( i < n \). I.e. the sequence has infinitely many distinct elements. Therefore the structure is infinite.