Problem 1. Determine if the following are tautologies:

(a) \((R \rightarrow (S \lor Q)) \lor (R \lor (S \rightarrow Q))\)
(b) \((R \leftrightarrow P) \lor (P \rightarrow \neg R)\)

Solution: Both (a) and (b) are tautologies. To see that in the case of (a), if \(R\) is false, then the first disjunct is true and if \(R\) is true, then the second disjunct is true. For part (b), note that the second disjunct is equivalent to \((\neg P \lor \neg R)\). I.e. if this fails, it means that both \(P\) and \(R\) are true and so the first disjunct holds.

Problem 2. Prove or refute the following:

(a) If \(\Sigma \models (\alpha \land \beta)\), then \(\Sigma \models \alpha\) and \(\Sigma \models \beta\)
(b) If \(\Sigma \models (\alpha \lor \beta)\), then \(\Sigma \models \alpha\) or \(\Sigma \models \beta\)

Solution:
(a) is true: argue for any truth assignment \(v\) such that \(v\) satisfies \(\Sigma\), we have that \(\bar{v}(\alpha \land \beta) = T\), and so it must be the case that \(\bar{v}(\alpha) = T\) and \(\bar{v}(\beta) = T\).
(b) is false: take \(\alpha = A\), \(\beta = \neg A\), and \(\Sigma = \emptyset\). Then argue that \(\Sigma \models (\alpha \lor \beta)\), but \(\Sigma \not\models \alpha\) and \(\Sigma \not\models \beta\).

Problem 3. Let \(S\) be the set of all sentence symbols, and let \(\check{S}\) be the set of all (sentential) formulas built up from \(S\). Fix a truth assignment \(\nu : S \rightarrow \{T, F\}\). Without using the recursion theorem, show that there is at most one extension \(\check{\nu} : \check{S} \rightarrow \{T, F\}\) satisfying the truth table conditions.

Solution: Let \(\nu_1, \nu_2\) be two extensions of \(\nu\) satisfying the truth table conditions. We will show that \(\nu_1(\alpha) = \nu_2(\alpha)\) for all \(\alpha\) by induction on formulas.

For the base case, if \(\alpha\) is a sentence symbol, then \(\nu_1(\alpha) = \nu(\alpha) = \nu_2(\alpha)\).

For the inductive case, there are two subcases:

1. \(\alpha = \neg \beta\). Then \(\nu_1(\alpha) = T\) iff \(\nu_1(\beta) = F\) iff \(\nu_2(\beta) = F\) (since by the inductive hypothesis \(\nu_1(\beta) = \nu_2(\beta)\)) iff \(\nu_2(\alpha) = T\).
2. \(\alpha = \beta \lor \gamma\). Then \(\nu_1(\alpha) = T\) iff \(\nu_1(\beta) = T\) or \(\nu_1(\gamma) = T\) iff \(\nu_2(\beta) = T\) or \(\nu_1(\gamma) = T\) (by the inductive hypothesis) iff \(\nu_2(\alpha) = T\).

So, by induction \(\nu_1 = \nu_2\)

Problem 4. Show that \(\{\land, \leftrightarrow, +\}\) is complete, but \(\{\land, +\}\) is not complete. Here \(\alpha + \beta\) means \((\alpha \lor \beta) \land \neg(\alpha \land \beta)\) i.e. either \(\alpha\) or \(\beta\) is true, but not both.
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Solution:
\( \neg \alpha \) is equivalent to \( \alpha \leftrightarrow (\alpha + \alpha) \). So, since \( \{ \neg, \land \} \) is complete, it follows that \( \{ \land, \leftrightarrow, + \} \) is complete.

To show that \( \{ \land, + \} \) is not complete, prove that for any formula using one sentence symbol, \( A \), and connective symbols among \( \{ \land, + \} \), this formula cannot be equivalent to \( \neg A \).

Problem 5. Show that \( \{ \bot, \rightarrow \} \) is complete, but \( \{ \land, \rightarrow \} \) is not complete.

Solution:
\( \neg \alpha \) is equivalent to \( \alpha \rightarrow \bot \). \( \alpha \lor \beta \) is equivalent to \( \neg \alpha \rightarrow \beta \). So, since \( \{ \neg, \lor \} \) is complete, it follows that \( \{ \bot, \rightarrow \} \) is complete.

To show that \( \{ \land, \rightarrow \} \) is not complete: for any formula using one sentence symbol, \( A \), and connective symbols among \( \{ \land, \rightarrow \} \), show that \( A \not\equiv \neg A \).

Problem 6. Show that \( \{ \rightarrow, + \} \) is complete, but \( \{ \leftrightarrow, + \} \) is not complete.

Solution:
\( \neg \alpha \) is equivalent to \( \alpha \rightarrow (\alpha + \alpha) \). \( \alpha \lor \beta \) is equivalent to \( \neg \alpha \rightarrow \beta \). So, since \( \{ \neg, \lor \} \) is complete, it follows that \( \{ +, \rightarrow \} \) is complete.

To show that \( \{ \leftrightarrow, + \} \) is not complete, prove that for any formula \( \phi \) with two sentence symbols, \( A, B \), and connectives among \( \{ \leftrightarrow, + \} \), we have that the number of truth assignments that satisfy \( \phi \) is even. Then no formula with these connectives is equivalent to \( A \land B \).

Problem 7. Recall that the corollary to the Compactness theorem states that if \( \Sigma \models \tau \), then there is some finite \( \Delta \subset \Sigma \) such that \( \Delta \models \tau \). Show that the Compactness theorem is equivalent to this corollary. (Prove both directions.)

Solution: For one direction, suppose the Compactness theorem holds and that \( \Sigma \models \tau \). Then \( \Sigma \cup \{ \neg \tau \} \) is not satisfiable and so by the compactness theorem, it is not finitely satisfiable. So there is some finite \( \Delta \subset \Sigma \) such that \( \Delta \cup \{ \neg \tau \} \) is not satisfiable. Then \( \Delta \not\models \tau \).

For the other direction, suppose that the corollary holds and that \( \Sigma \) is finitely satisfiable. We have to show that \( \Sigma \) is satisfiable. Let \( \tau \in \Sigma \) and let \( \Sigma^* = \Sigma \setminus \{ \tau \} \).

Claim 1. For any finite \( \Delta \subset \Sigma^* \), \( \Delta \not\models \neg \tau \).

Proof. Fix a finite \( \Delta \subset \Sigma^* \). Since \( \Sigma \) is finitely satisfiable, we have that \( \Delta \cup \{ \tau \} \) is satisfiable. Therefore, \( \Delta \not\models \neg \tau \).

By the claim and the corollary, it follows that \( \Sigma^* \not\models \neg \tau \). So \( \Sigma^* \cup \{ \tau \} = \Sigma \) is satisfiable.
Problem 8. Suppose $\Sigma$ is satisfiable and complete (i.e. for every formula $\phi$, either $\phi \in \Sigma$ or $\neg \phi \in \Sigma$). Define a truth assignment $v$ by $v(A) = T$ iff $A \in \Sigma$. Show that for each $\phi \in \Sigma$, $\bar{v}(\phi) = T$ iff $\phi \in \Sigma$.

Solution: by induction on $\phi$

Problem 9. Write the following in the first order language of set theory, $\mathcal{L} = \{\in\}$, as follows: first write down the sentence using symbols among $\{\neg, \vee, \wedge, \to, \leftrightarrow, \forall, \exists, \in\}$ and variables. Then rewrite the expression using only symbols among $\{\neg, \to, \in\}$ and variables (unless of course the already written expression is in this form).

(a) $x$ is a subset of $y$:
\[ \forall z (z \in x \to z \in y) \]

(b) The power set of $x$ is equal to the power set of $y$. (The power set of $x$, denoted $P(x)$, is the set of all subsets of $x$):
\[ \forall v (v \in z \leftrightarrow v \in y) \]

(c) The well foundedness axiom: every nonempty $x$ has an element $y$, such that no other element of $x$ belongs to $y$:
\[ \forall x (\exists y (y \in x) \to \exists z (z \in x \to \neg (z \in y))) \]

Problem 10. Let $P$ be a two-place predicate. Consider the formula:

\[ \phi = \forall x \exists y P(x, y) \to \exists y \forall x P(x, y) \]

(1) Give an example of a model in which $\phi$ is false.

(2) Give an example of a model in which $\phi$ is true.

(Note that when you describe the model, you have to say what is the interpretation of $P$.)

Solution: For part (1), let $\mathfrak{A} = (\mathbb{R}, <)$. Here $P^{\mathfrak{A}} = <$. Then $\mathfrak{A} \not\models \forall x \exists y P(x, y)$ but $\mathfrak{A} \models \exists y \forall x P(x, y)$, so $\mathfrak{A} \not\models \phi$.

For part (2), let $\mathfrak{A} = (\mathbb{N}, \geq)$. Here $P^{\mathfrak{A}} = \geq$. Then $\mathfrak{A} \models \exists y \forall x P(x, y)$ (take $y$ to be 0), so $\mathfrak{A} \models \phi$. 