THE FAILURE OF THE SINGULAR CARDINAL HYPOTHESIS AND SCALES

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Abstract. Starting from a supercompact cardinal \( \kappa \), we build a model, in which \( \kappa \) is singular string limit, the singular cardinal hypothesis fails at \( \kappa \) and there are no very good scales at \( \kappa \). Moreover there is a bad scale at \( \kappa \), and so weak square fails.

1. Introduction

The Singular Cardinal Problem is the problem to find a complete set of rules for the behavior of the operation \( \kappa \mapsto 2^{\kappa} \) for singular cardinals \( \kappa \). One central theme is how much “reflection-type” properties are consistent with the failure of the singular cardinal hypothesis (SCH). SCH states that if \( \kappa \) is singular and \( 2^{\text{cf}(\kappa)} < \kappa \), then \( \kappa^{\text{cf}(\kappa)} = \kappa^+ \). In particular, if \( \kappa \) is strong limit singular, then \( 2^\kappa = \kappa^+ \).

Scales are a central concept in PCF theory. Given a singular cardinal \( \kappa = \sup_n \kappa_n \), where each \( \kappa_n \) is regular, a scale of length \( \kappa^+ \) is a sequence of functions \( \langle f_\alpha \mid \alpha < \kappa^+ \rangle \) in \( \prod_n \kappa_n \) that is increasing and cofinal with respect to the eventual domination ordering. A point \( \alpha < \kappa^+ \) with \( \text{cf}(\alpha) > \omega \) is good if there is an unbounded \( A \subseteq \alpha \) such that \( \{ f_\beta(n) \mid \beta \in A \} \) is strictly increasing for all large \( n \). If \( A \) is a club in \( \alpha \), then \( \alpha \) is very good. A scale is good, resp. very good, if on a club every point of uncountable cofinality is good, resp. very good. A scale is bad if it is not good.

Very good scales follow from intermediate square principles, and in turn imply failure of simultaneous stationary reflection (Cummings-Foreman-Magidor, [2]). Thus the nonexistence of a very good scale is a “reflection-type” property, and it has been open whether it is consistent with the failure of SCH at a singular strong limit cardinal.

Extender based forcing, developed by Gitik and Magidor [5], violates SCH at a singular cardinal \( \kappa \) while keeping GCH below \( \kappa \). The set up is to start with a singular \( \kappa \), such that \( \kappa = \sup_n \kappa_n \), each \( \kappa_n \) is a strong cardinal, and then force to add many sequences though \( \kappa \), but without adding bounded subsets at \( \kappa \). In his Ph.D. thesis [7], Assaf Sharon modified this forcing to construct a model, where SCH fails at \( \kappa \) and there are no very good scales at \( \kappa \). In his model, however, bounded subsets of \( \kappa \) are added, and \( \kappa \) is no longer strong limit. More precisely, only \( \kappa_0 \) remains (regular) strong limit.

Another way to violate SCH is via Magidor’s supercompact Prikry forcings, [6]. An important variation is Gitik-Sharon’s diagonal supercompact Prikry, [4]. In [8], we defined a forcing notion, called hybrid Prikry, which combines the diagonal supercompact forcing from Gitik-Sharon [4] and the original extender based forcing. This poset simultaneously singularizes all cardinals in the interval \([\kappa, \kappa^{+\omega})\), for a large cardinal \( \kappa \), and uses extenders to add many Prikry sequences to \( \prod_n \kappa_n \), so that SCH is violated. Here we define a modified hybrid Prikry forcing, combining ideas from [8] and the modified extender based forcing.
from Assaf Sharon’s thesis, [7]. We use it to obtain the consistency of not SCH and no very good scale at a singular strong limit.

Our main forcing does not add bounded subsets of κ, thereby keeping it strong limit. However, before the main forcing, we need some Laver type preparation, to achieve no very good scales. Thus we can’t quite keep GCH below κ, but we do have GCH at every inaccessible α < κ.

**Theorem 1.1.** Starting from a supercompact, it is consistent that SCH fails at a strong limit κ, and there is no very good scale at κ.

In our forcing extension, there is also a bad scale:

**Theorem 1.2.** Starting from a supercompact, it is consistent that SCH fails at a strong limit κ, there is a bad scale at κ and for all inaccessible cardinals α < κ, 2^α = α^+.

The existence of a bad scale implies that weak square fails in our model. The failure of SCH together with a bad scale was already achieved in [4] at \( \aleph_\omega \). However, there is no natural way to modify their forcing for \( \aleph_\omega \) together with a bad scale was already achieved in \([4]\) at \( \aleph_\omega \).

In section 2 we define the forcing. Preservation of \( \mu \) is shown in section 3, and the Prikry lemma is given in section 4. In section 5, we show that SCH fails in the generic extension. Section 6 has the proof that there is no very good scale in the generic extension, and in section 7 we define a bad scale.

### 2. The Forcing

Suppose that GCH holds; let \( \kappa \) be supercompact, and let \( \langle \kappa_n \mid n < \omega \rangle \) be an increasing sequence of strong cardinals above \( \kappa \). Denote \( \kappa_\omega = \sup_n \kappa_n \), \( \mu := \kappa_\omega^+ \). For each \( n < \omega \), let \( U_n \) be a normal measure on \( P_\kappa(\kappa_n) \), and let \( j_n := j_{U_n} \).

Suppose also that \( Ult(U_n) = \kappa_n \) is \( j_n(\mu^+) \)-strong. So for a \( U_n \)-measure one set of \( x \)'s in \( P_\kappa(\kappa_n) \), \( \kappa_n^+ := o.t(x) \) is a \( \mu^+ \)-strong cardinal. Say this is witnessed by \( j_x : V \rightarrow M_x \).

Let \( x \in P_\kappa(\kappa_n) \) be as above. Let \( \langle E_{x\alpha} \mid \alpha < \mu^+ \rangle \) be \( \kappa_n^+ \) complete ultrafilters on \( \kappa_n^+ \), where \( E_{x\alpha} = \{ Z \subseteq \kappa_n^+ \mid \alpha \in j_x(Z) \} \). Arguing as in \([3]\) we define a strengthening of the Rudin-Keisler order: for \( \alpha, \beta < \mu^+ \), set \( \alpha \leq_E \beta \) if \( \alpha \leq \beta \) and there is a function \( f : \kappa_n^+ \rightarrow \kappa_n^+ \), such that \( j_x(f)(\beta) = \alpha \). For \( \alpha \leq_E \beta \), fix projections \( \pi_{\beta\alpha} : \kappa_n^+ \rightarrow \kappa_n^+ \) to witness this ordering, setting \( \pi_{\alpha,\alpha} \) to be the identity. We do this as in Section 2 of \([3]\) with respect to \( \kappa_n^+ \), so that we have:

1. \( j_x \pi_{\beta\alpha}(\beta) = \alpha \).
2. For all \( a < \mu^+ \) with \( |a| < \kappa_n^+ \), there are unboundedly many \( \beta < \mu^+ \), such that \( \alpha <_E \beta \) for all \( \alpha \in a \).
3. For \( \alpha < \beta \leq \gamma \), if \( \alpha \leq_E \gamma \) and \( \beta \leq_E \gamma \), then \( \{ \nu < \kappa_n^+ \mid \pi_{\gamma,\alpha}(\nu) < \pi_{\gamma,\beta}(\nu) \} \in E_{x\gamma} \).
4. If \( \{ i < \tau \} \subseteq \alpha < \mu^+ \) with \( \tau < \kappa \), are such that for all \( i < \tau \), \( \alpha_i <_E \alpha \), then there is \( A \in E_{x\alpha} \), such that for all \( \nu \in A \), for all \( i, j < \tau \), if \( \alpha_i \leq_E \alpha_j \), then \( \pi_{\alpha,\alpha_i}(\nu) = \pi_{\alpha_j,\alpha_i}(\pi_{\alpha,\alpha_j}(\nu)) \).
2.1. The modules.

**Definition 2.1.** The poset \(Q_{x}^{n} = Q_{x0}^{n} \cup Q_{x1}^{n}\) is defined as follows:
\(Q_{x1}^{n} = \{ f : \mu^{+} \to \kappa^{n}_{x} \mid f \leq \kappa^{n}_{x} \}\) and \(\leq_{x1}\) is the usual ordering. \(Q_{x0}^{n}\) has conditions of the form \(p = \langle a, A, f \rangle\) such that:

- \(a < \mu^{+}, |a| < \kappa^{n}_{x}\), for all \(\beta \in a, \beta \leq E_{x} \max(a)\),
- \(f \in Q_{x1}^{n}\), and \(a \cap \text{dom}(f) = \emptyset\)
- \(A \in E_{x} \max(a)\),
- for all \(\alpha \leq \beta \leq E_{x}\gamma\) in \(a\), for all \(\nu \in \pi_{\max a, \gamma}(\nu)\), \(\pi_{\gamma, \alpha}(\nu) = \pi_{\gamma, \beta}(\nu)\).

\(\langle b, B, g \rangle \leq_{x0} \langle a, A, f \rangle\) if:

1. \(b \supset a\),
2. \(\pi_{\max b, \max a}(B) \subset A\),
3. \(g \supset f\).

Define \(\leq_{x}^{*} = \leq_{x0} \cup \leq_{x1}\) and for \(p, q \in Q_{x}^{n}\), \(p \leq_{x}^{*} q\), if \(p \leq_{x} q\) or \(p \in Q_{x1}^{n}\), \(q = \langle a, A, f \rangle \in Q_{x0}^{n}\) and

1. \(p \supset f, a \subset \text{dom}(p)\)
2. \(p(\max(a)) \in A\)
3. for all \(\beta \in a, p(\beta) = \pi_{\max a, \beta}(p(\max(a)))\).

**Definition 2.2.** For a condition \(p = \langle a, A, f \rangle \in Q_{x0}^{n}\) and \(\nu \in A\), let \(p^{\perp}\nu = f \cup \{\langle \beta, \pi_{\max a, \beta}(\nu) \rangle \mid \beta \in a\}\). I.e. \(p^{\perp}\nu\) is the weakest extension of \(p\) in \(Q_{x1}^{n}\) with \(\nu\) in its range.

Note that if \(g \in Q_{x1}^{n}\), with \(g \leq p\), there is a unique \(\nu \in A\) such that \(g \leq p^{\perp}\nu\) (take \(\nu = g(\max(a))\)).

Finally, for \(n < \omega\), denote
\[Q_{x}^{n} := [x \mapsto Q_{x0}^{n}]_{U_{n}}, Q_{x}^{1} := [x \mapsto Q_{x1}^{n}]_{U_{n}},\text{ and } Q_{x}^{n} := [x \mapsto Q_{x1}^{n}]_{U_{n}}.\]

Since each \(Q_{x0}^{n}\) is \(\kappa^{x}_{n}\)-closed, we have that \(Q_{x0}^{n}\) is \(\kappa^{x}_{n}\)-closed. Also, for each \(\alpha = [x \mapsto \alpha_{x}]_{U_{n}} < j_{n}(\mu^{+})\), set \(E_{\alpha}^{n} := [x \mapsto E_{x, \alpha_{x}}]_{U_{n}}\). \(Q^{n}\) is the extender based module over \(\kappa_{n}\) with respect to \(\langle E_{\alpha}^{n} \mid \alpha < j_{n}(\mu^{+}) \rangle\) and Cohen parts of size less than or equal to \(\kappa_{n}\).

2.2. The main forcing. For \(x, y \in \mathcal{P}_{\kappa_{\omega}}(\kappa_{\omega})\), we will denote \(\kappa_{x} = \kappa \cap x\) and use the notation \(x < y\) to mean \(x \subset y\) and \(o.t.(x) < \kappa_{y}\). Since on a measure one set, \(\kappa_{x}\) is an inaccessible cardinal, we assume this is always the case. Similarly, for each \(k\), on a measure one set, for \(x \in \mathcal{P}_{\kappa}(\kappa_{k})\), \(\kappa_{k}^{+} = o.t.(x)\) is strong. So we assume this is always the case, too.

**Definition 2.3.** Suppose we have sets \(A_{i} \in U_{i}, B_{i} \in [x \mapsto E_{x, \alpha_{x}}]_{U_{i}}\) where each \(\alpha_{x} \in \mu^{+}\).

Let \(\prod_{j \geq 1} A_{i} \times B_{i}^{< \omega}\) denote the set of all finite sequences \(\langle \vec{x}, \vec{y} \rangle\), where for some \(n\),

1. \(\vec{x} = \langle x_{1}, \ldots, x_{n} \rangle\), is such that each \(x_{i} \in A_{i}\) and \(x_{1} < x_{i+1} < \ldots < x_{n}\),
2. \(\vec{y} = \langle \nu_{1}, \ldots, \nu_{n} \rangle\), is increasing and such that each \(\nu_{i} \in B_{i}\),
3. each \(\nu_{i} \in x_{i}\).

For every \(i, B_{i} \in [x \mapsto E_{x, \alpha_{x}}]_{U_{i}}\), and \(\nu \in B_{i}\) as above, fix representative functions \(x \mapsto \nu_{x}\), such that \(\nu := [x \mapsto \nu_{x}]_{U_{i}}\).
Definition 2.4. Conditions in \( \mathbb{P} \) are of the form
\[
p = \langle x_0, f_0, ..., x_{l-1}, f_{l-1}, A_l, F_l, A_{l+1}, F_{l+1}, ... \rangle
\]
where \( l = \text{lh}(p) \) and:

1. For \( n < l \), \( x_n \in \mathcal{P}_\kappa(\kappa_n) \), and for \( i < n \), \( x_i < x_n \).
2. For \( n \geq l \), \( A_n \subseteq U_n \), and \( x_{l-1} < y \) for all \( y \in A_l \).
3. For \( n \geq l \), \( \text{dom}(F_n) = A_n \), and for \( y \in A_n \), \( F_n(y) = \langle a_n(y), A_n(y), f_n(y) \rangle \), where \( a_n(y) \in [\mu^+]^{<\kappa^n} \), \( A_n(y) \in E_{y, \text{max}(a_n(y))} \).
   Denote \( B_n := [y \mapsto A_n(y)]_{U_n} \).
4. For \( n < l \), \( \text{dom}(f_n) = \mathcal{P}_\kappa(\kappa_n) \), for every \( x \in \mathcal{P}_\kappa(\kappa_n) \),
   \[
f_n(x) : \prod_{i \geq 1} A_i \times B_i \langle \omega \rangle \cap \{ (\bar{z}, \bar{\nu}) \mid x < \bar{z} \} \rightarrow \mathbb{Q}_{21}^n,
\]
   and for \( \langle \bar{z}, \bar{\nu} \rangle \subseteq \langle \bar{z}', \bar{\nu}' \rangle \), \( f_n(x)(\langle \bar{z}', \bar{\nu}' \rangle) \leq f_n(x)(\langle \bar{z}, \bar{\nu} \rangle) \).
5. For \( n \geq l \), \( F_n(y) = \langle a_n(y), A_n(y), f_n(y) \rangle \), such that:
   (a) \( \text{dom}(f_n(y)) = \prod_{i > n} A_i \times B_i \langle \omega \rangle \cap \{ (\bar{z}, \bar{\nu}) \mid y < \bar{z} \} \) and for each \( (\bar{z}, \bar{\nu}) \in \text{dom}(f_n(y)) \),
   \[
   \langle a_n(y), A_n(y), f_n(y)(\langle \bar{z}, \bar{\nu} \rangle) \rangle \in \mathbb{Q}_{y_0}^n,
   \]
   (b) if \( \langle \bar{z}, \bar{\nu} \rangle \subseteq \langle \bar{z}', \bar{\nu}' \rangle \), \( f_n(y)(\langle \bar{z}', \bar{\nu}' \rangle) \leq f_n(y)(\langle \bar{z}, \bar{\nu} \rangle) \).
6. For \( l \leq n < m \), \( y \in A_n \), \( y' \in A_m \), \( y < y' \), we have \( a_n(y) \subseteq a_m(y') \).

For a condition \( p \) as above we will use \( f_n^p, x_n^p, n < \text{lh}(p) \) and \( A_n^p, B_n^p, F_n^p, F_n^p(y) = \langle a_n^p(y), A_n^p(y), f_n^p(y) \rangle, n \geq \text{lh}(p) \) to denote its components as defined above. Also for \( n \geq \text{lh}(p) \), let \( \beta^p_n := [x \mapsto \max(\langle a_n^p(x) \rangle)]_{U_n} \).

We say that \( q \leq^* p \) if \( \text{lh}(q) = \text{lh}(p) = l \), and

1. for all \( n < l \), \( x_n^q = x_n^p \) and for \( n \geq l \), \( A_n^q \subseteq A_n^p \),
2. for all \( n \geq l \), \( y \in A_n^q \), \( a_n^q(y) \supseteq a_n^p(y), \pi_{\text{max}(a_n^q(y)}, a_n^p(y)) = A_n^q(y) \subseteq A_n^p(y) \).

For \( n \geq l \) and \( \bar{\nu} \in \prod_{i \leq k} B_i^p \), denote \( \pi(\bar{\nu}) = \langle \pi_{\beta^p_0, \beta^p_0}(\nu_0), ..., \pi_{\beta^p_{k-1}, \beta^p_k}(\nu_k) \rangle \).

3. for all \( n < l \), \( \forall_{U_n} x \), for all \( \langle \bar{z}, \bar{\nu} \rangle \in \text{dom}(f_n^p(x)) \), \( f_n^q(x)(\langle \bar{z}, \bar{\nu} \rangle) \leq f_n^p(x)(\langle \bar{z}, \pi(\bar{\nu}) \rangle) \),
4. for all \( n \geq l \), \( y \in A_n^q \) and \( \langle \bar{z}, \bar{\nu} \rangle \in \text{dom}(f_n^q(y)) \),
   \[
   \langle a_n^q(y), A_n^q(y), f_n^q(y)(\langle \bar{z}, \bar{\nu} \rangle) \rangle \leq \mathbb{Q}_{y_0}^n \langle a_n^p(y), A_n^p(y), f_n^p(y)(\langle \bar{z}, \pi(\bar{\nu}) \rangle) \rangle.
   \]
5. for all \( n \geq l \), \( A_n^q \subseteq \{ x \mid \nu \in x \rightarrow \pi_{\beta^p_0, \beta^p_0}(\nu) \in x \} \). This is needed to ensure transitivity of \( \leq_{\mathbb{P}} \).

Suppose \( p \) has length \( l \), \( k > l \), and \( \langle \bar{x}, \bar{\nu} \rangle \in \prod_{i \geq 1} A_i^p \times B_1^p \langle \omega \rangle ; \bar{x} := \langle x_1, ..., x_{k-1} \rangle, \bar{\nu} := \langle \nu_1, ..., \nu_{k-1} \rangle \). We define the weakest \( k - l \)-step extension of \( p \) obtained from \( \langle \bar{x}, \bar{\nu} \rangle \) denoted by \( p \rightarrow \langle \bar{x}, \bar{\nu} \rangle \) to be the condition
\[
\langle x_0^p, f_0, ..., x_{l-1}^p, f_{l-1}, x_1, f_1, ..., x_{k-1}, f_{k-1}, A_k, F_k, A_{k+1}, F_{k+1}, ... \rangle,
\]
such that:

1. for \( n \geq k \), \( A_n = A_n^p \cap \{ y \mid x_{k-1} < y \} \),
2. for \( n < l \), for \( x \in \mathcal{P}_\kappa(\kappa_n) \), and \( \langle \bar{z}, \bar{\nu} \rangle \in \text{dom}(f_n(x)) \), \( f_n(x)(\langle \bar{z}, \bar{\nu} \rangle) = f_n^p(x)(\langle \bar{x}, \bar{\nu} \rangle \rightarrow \langle \bar{z}, \bar{\nu} \rangle) \).
(3) for \(l < n < k\), for \(x \in P_\kappa(\kappa_n)\) with \(\nu_x \in A^p_n(x)\), for all \(\langle z, \delta \rangle \in \text{dom}(f_n(x))\),
\[
f_n(x)((z, \delta)) = \langle a^p_n(x), A^p_n(x), f^p_n(x)(x_{n+1}, \ldots, x_{k-1}, \nu_{n+1}, \ldots, \nu_{k-1})(z, \delta)) \rangle - \nu_x;
\]
otherwise, if \(\nu_x \notin A^p_n(x)\), for all \(\langle z, \delta \rangle \in \text{dom}(f_n(x))\), set \(f_n(x)((z, \delta)) = \emptyset\).

(4) for \(n \geq k\) and \(y \in A_n\), we have \(F_n(y) = F^p_n(y)\).

We can finally define the full ordering:

**Definition 2.5.** \(q \leq p\) if \(q \leq^* p\) or for some \(\langle \bar{y}, \bar{v} \rangle\), we have that \(q \leq^* p - \langle \bar{y}, \bar{v} \rangle\).

3. Preservation of \(\mu\)

Define \(P_n := \{p \in P \mid \text{lh}(p) = n\}\). We will show that \((P_0, \leq^*)\) preserves \(\mu\). The idea will be that for every \(n\), we can regard it as a combination of two subposets, one with \(\kappa_n^{++}\)-c.c., and the other \(\kappa_{n+1}\)-closed. We use this to show that \((P_0, \leq^*\) preserves \(\kappa_n\) for every \(n\), and then conclude that it must preserve \(\mu\). We remark that our arguments can be adapted to show \((P_k, \leq^*)\) preserves \(\mu\), for every \(k < \omega\).

**Definition 3.1.** For \(p, q \in P_0\), we say that \(p \sim q\) if for all \(k < \omega\), \(B^p_k = B^q_k = B_k\), and there are measure one sets \(A_k \subset A^p_k \cap A^q_k\), such that for all \(k < \omega, x \in A_k, \langle z, \bar{v} \rangle \in \prod_{i<k} A_i \times B_i\)\( <^\omega\) with \(x < z\), we have that \(a^p_k(x) = a^q_k(x), A^p_k(x) = A^q_k(x), f^p_k(x)(z, \bar{v}) = f^q_k(x)(z, \bar{v}).\) Define \(p \leq^* q\) if there is \(p' \sim p\) with \(p' \leq q\).

Let \(\mathcal{P}_0 \upharpoonright [n, \omega) := \{A^p_n, F^p_n, A^p_{n+1}, F^p_{n+1}, \ldots \mid p \in \mathcal{P}_0\}\) with the induced ordering from \(\leq^*\) (which we denote the same). Note that \((\mathcal{P}_0, \leq^*)\) and \((\mathcal{P}_0, \leq^\sim)\) are isomorphic.

**Proposition 3.2.** \((\mathcal{P}_0 \upharpoonright [n, \omega), \leq^\sim)\) is \(\kappa_n\)-closed, for all \(n \geq 0\).

**Proof.** Suppose that \(\tau < \kappa_n\) and \(\langle p_\eta \mid \eta < \tau \rangle\) is a \(\leq^\sim\)-decreasing sequence in \(\mathcal{P}_0 \upharpoonright [n, \omega)\).

For each \(\eta < \delta\), let \(\langle A^\eta_\delta, n < k < \omega, x \in A^\eta_\delta(x)\rangle\) be measure one sets in \(U_k\), respectively, witnessing that \(p^\delta \leq^\sim p^\eta\).

For \(k \geq n\), set \(A^p_k = \bigtriangleup A^\eta_\delta = \{x \mid x \in \bigcap_{\eta \in x, \delta \in x} A^\eta_\delta\}\). Let \(\bar{m} < \mu^+\) be above the supremum of all of the domains of the \(f^p_k\)'s, i.e.

\[
\bar{m} > \sup_{\eta < \tau, k < \omega, x \in A^p_k(x), \langle z, \bar{v} \rangle \in \text{dom}(f^p_k(x))} \text{dom}(f^p_k(x)(z, \bar{v})).
\]

Inductively on \(k\), for all \(x \in A^p_k\), set

\[
a^p_k(x) = \bigcup_{\eta \in x \cap \tau} a^p_k(x) \cup \bigcup_{n < m < k, w \in A^p_m} a^p_m(w) \cup \{m\},
\]

where \(m\) is a maximal element above \(\bar{m}\). Then let \(A^p_k(x) = \bigcap_{\eta \in x \cap \tau} \pi^{-1}_{\eta}(A^p_k(x))\), where

\[
\pi^p_k = \pi_{\max(a^p_k(x)), \max(a^p_k(x))}.
\]

Now let \(B' := \{y \mapsto A^p_k(x)|y|u_i\}\). For all \(\langle z, \bar{v} \rangle \in \prod_{i<k} A^p_i \times B'_i <^\omega\) with \(x < z\), define

\[
f^p_k(x)(z, \bar{v}) = \bigcup_{\eta \in x \cap \tau} f^p_k(x)(\pi^\eta(z, \pi^\eta(\bar{v}))).
\]

where \(\pi^\eta\) is the corresponding pointwise projections from the maximal coordinates of \(p\) to the maximal coordinates of \(p_\eta\).

We claim that \(p\) is as desired. For if \(\eta < \tau\), for \(k \geq n\), let \(A_k = A^p_k \cap \{x \mid \eta \in x\} \cap \{x \mid \nu \in x \rightarrow \pi^p_k, \pi^\eta(\nu) \in x\}\). Then \(\langle A_k \mid k \geq n \rangle\) witness that \(p \leq^\sim p_\eta\). 

\(\square\)
For \( n > 0 \) and \( p \in \mathbb{P}_0 \), let \( \pi_n(p) = (A^p_0, F^p_0, A^p_1, F^p_1, \ldots, A^p_{n-1}, F^p_{n-1}) \). Set \( \mathbb{P}_n := \{ \pi_n(p) \mid p \in \mathbb{P}_0 \} \) with the natural induced ordering from \( \leq^* \).

**Proposition 3.3.** For all \( n \geq 0 \), \( \mathbb{P}_{0n+1} \) has the \( \kappa_n^{++} \) c.c.

**Proof.** By induction on \( n \). Suppose for contradiction that \( \{ \pi_{n+1}(p_\eta) \mid \eta < \kappa_n^{++} \} \) is an antichain in \( \mathbb{P}_{0n+1} \). By strengthening each \( p_\eta \) if necessary, we may assume that the part above \( n \) is the same, i.e. for all \( i > n \), \( [F^i_1]_{U_i} = [F^i_1]_{U_i} \) for all \( \eta \). For \( i > n \), denote \( [F^i_1]_{U_i} := \langle a^*_i, B_i, f^*_i \rangle \), and let \( \alpha_i = \max(a^*_i) \). Then each \( B_i \in E_i, \alpha_i \). For \( m > n \), set \( i_m := j_{E_{n+1,\alpha_{n+1}}} \circ j_{n+1} \circ \ldots \circ j_{E_m,\alpha_m} \circ j_m \).

Fix \( x \in A^n_0 \). We will define functions \( f^n_{x,m} \) for \( n < m \) as follows.

- If \( m = n+1 \), for \( \nu \in B_{n+1} \), let \( f^n_{x,\nu,\nu,n+1} := [z \mapsto f^{\nu}_n((z, \nu))]_{U_{n+1}} \). Then each \( f^n_{x,\nu,\nu,n+1} \)|\( \leq \kappa_x^n \). So, by applying the \( \Delta \)-system lemma, we get an unbounded \( \nu \) and \( f^n_{x,\nu,\nu,n+1} \) of size less than or equal to \( \kappa_x^n \).

- If \( m = n+2 \), for \( \nu \in B_{n+2} \), let \( f^n_{x,\nu,\nu,n+2} := [z \mapsto f^{\nu}_n((z, \nu))]_{U_{n+2}} \). Then each \( f^n_{x,\nu,\nu,n+2} \)|\( \leq \kappa_x^n \). So, by applying the \( \Delta \)-system lemma, we get an unbounded \( \nu \) and \( f^n_{x,\nu,\nu,n+2} \) of size less than or equal to \( \kappa_x^n \).

- Continue in a similar fashion for all \( m > n \).

Then each \( f^n_{x,m} \) is a partial function from \( i_m(\mu^+) \) to \( \kappa_x^n \) of size less than or equal to \( \kappa_x^n \). Define a partial function \( F^n_m : \mathcal{P}_\kappa(\kappa_n) \times i_m(\mu^+) \rightarrow \{ Y \} \cup \kappa \) by:

\[
F^n_m(x, \alpha) := \begin{cases} 
Y & \text{if } \alpha \in i_m(a^n_\alpha(x)) \\
 f^n_{x,m}(\alpha) & \text{if } \alpha \in \text{dom}(f^n_{x,m}) \end{cases}
\]

Let \( F^n \) be the function given by \( F^n(m, x, \alpha) = F^n_m(x, \alpha) \). This is a function of size less than \( \kappa_x^n \). So, by applying the \( \Delta \)-system lemma, we get an unbounded \( I \subset \kappa_n^{++} \), such that \( \langle F^n \mid \eta \in I \rangle \) forms a \( \Delta \) system, and the functions have the same value on the kernel. Note that this implies that for all \( \eta, \delta \) in \( I \) and for all \( n < m, x \in \mathcal{P}_\kappa(\kappa_n), i_m(a^n_\alpha(x)) \cap \text{dom}(f^n_{x,m}) = \emptyset \).

By the inductive hypothesis, if \( n > 0 \), \( \mathbb{P}_{0n} \) has the \( \kappa_{n-1}^{++} \) c.c. So let \( \eta, \delta \) be distinct points in \( I \), such that if \( n > 0, \pi_n(p_\eta) \) and \( \pi_n(p_\delta) \) are compatible. We will construct \( p \in \mathbb{P}_0 \), such that \( \pi_{n+1}(p) \) is a common extension of \( \pi_{n+1}(p_\eta) \) and \( \pi_{n+1}(p_\delta) \).

Let \( \bar{m} < \mu^+ \) be above the supremum of the domains of \( f^{\eta}_k(x)(h) \) and \( f^{\delta}_k(x)(h) \), for \( k \leq n, x \in A^n_k \cap A^n_{\bar{m}}, h \in \text{dom}(f^{\eta}_k(x)) \cap \text{dom}(f^{\delta}_k(x)) \). Also, let \( r \) be a common extension of \( \pi_n(p_\eta) \) and \( \pi_n(p_\delta) \), such that for all \( k < n, x \in A^n_k, a^\eta_k(x) = a^\eta_k(x) \cup a^\delta_k(x) \cup c, \) where \( c \subset \mu^+ \setminus \bar{m} \). We will define \( p \) so that \( p \upharpoonright n \sim r \).

For \( i < n \), set \( A^i_k = A^i_k \), for \( x \in A^n_k \), set \( a^p_k(x) = a^\eta_k(x) \cup a^\delta_k(x) \cup A^i_k(x) \). And then \( B^p_i = B^i_k \).

Also set \( A^n_0 = A^n_{\bar{m}} \cap A^n_{\bar{m}} \). For \( x \in A^n_0 \), let

\[
a^n_p(x) = a^n_p(x) \cup a^n_p(x) \cup \bigcup_{i < n, w \in A^n_i, w < x} a^n_p(w) \cup \{ m' \},
\]
where $m' > m$ is a maximal element in the extender ordering. Then, set

$$A^p_n(x) = \pi^{-1}_{m',m_q} \left( A^p_n(x) \right) \cap \pi^{-1}_{m',m_\delta} \left( A^s_n(x) \right),$$

where $m_q, m_\delta$ are the maximal elements of $a^p_n(x)$ and $a^s_n(x)$ respectively. Finally, for all $m > n$, let

$$f_{x,m} = f_{x,m}^p \cup f_{x,m}^s.$$

This is a well-defined function because the values on the kernel of the $\Delta$ system obtained above are the same.

Denote:

- $f_{x,m} = \left[ \nu \mapsto f^m_n(x)(\nu) \right]_{E_{n+1},\alpha_{n+1}}$;
- $f^m_n(x)(\nu) = \left[ y \mapsto f^m_n(x)(\nu)(y) \right]_{U_{n+1}}$;
- $f^m_n(x)(\nu)(y) = \left[ \delta \mapsto f^m_n(x)(\nu)(y)(\delta) \right]_{E_{n+2},\alpha_{n+2}}$;
- $f^m_n(x)(\nu)(y)(\delta) = \left[ z \mapsto f^m_n(x)(\nu)(y)(\delta)(z) \right]_{U_{n+2}}$;
- ... and so on until we reach $m$.

Then we have that:

$$\forall^*_{E_{n+1},\alpha_{n+1}} \nu_{n+1} \forall^*_{U_{n+1},y_{n+1}} \forall^*_{E_{n+2},\alpha_{n+2}} \nu_{n+2} \forall^*_{U_{n+2},y_{n+2}} \forall^*_{E_{m},\alpha_{m}} \nu_{m} \forall^*_{U_{m},y_{m}}$$

(\dagger) : $f^m_n(x)(\nu_{n+1}(y_{n+1})...\nu_{m}(y_{m}) = f^m_n(x)(y_{n+1},...,y_{m},\nu_{n+1},...\nu_{m}) \cup f^m_n(x)(\nu_{n+1},...,y_{m},\nu_{n+1},...\nu_{m})$

and

$$\text{dom} \left( f^m_n(x)(\nu_{n+1}(y_{n+1})...\nu_{m}(y_{m})) \right) \cap a^p_n(x) = \emptyset.$$

Then by taking diagonal intersection, for all $x \in A^p_k \cap A^p_s$, for all $m > n$, we have measure one sets $A^x_{n+1}, A^x_{n+2},..., A^x_{m}$ and $B^x_{n+1}, B^x_{n+2},..., B^x_{m}$, where each $A^x_{i} \subseteq U_i$, $B^x_{i} \in E_{i,\alpha_i}$, such that for all $\langle y, y \rangle \in \prod_{n<i \leq m} A^x_{i} \times B^x_{i}$ with $x < y$, we have that the above equality holds.

We illustrate how these sets are defined for $m = n + 2$:

- $B_{n+1} = \{ \nu \mid \forall^* y, \forall^* z (\dagger) \text{ holds for } \langle y, z, \nu, \delta \rangle \}$.

- For every $\nu \in B_{n+1}$, let $A_{\nu} \subseteq U_{n+1}$ witness it.

  Set $A_{n+1} = \Delta A_{\nu} = \{ y \mid y \in \bigcap_{\nu \in \nu} A_{\nu} \} \subseteq U_{n+1}$;

- For all $\nu \in B_{n+1}$, for all $y \in A_{\nu}$, let $B_{\nu,y} \subseteq E_{n+2,\alpha_{n+2}}$ witness it.

  Set $B_{n+2} = \bigcap_{\nu \in B_{n+1}} B_{\nu,y} \subseteq E_{n+2,\alpha_{n+2}}$.

- For all $\nu \in B_{n+1}, y \in A_{\nu}$, and $\delta \in B_{\nu,y}$, let $A_{\nu,y,\delta} \subseteq U_{n+2}$ witness it.

  Set $A_{n+2} := \Delta A_{\nu,y,\delta} = \{ z \mid z \in \bigcap_{\delta \in B_{\nu,y}} A_{\nu,y,\delta} \} \subseteq U_{n+2}$.

For such $x$, for $i > n$, let

$$A^x_{i} = \bigcap_{i \leq m < \omega} A^x_{i} \cap B^x_{i} = \bigcap_{i \leq m < \omega} B^x_{i}.$$

Then set $A_i = \Delta A^x_{i}, B^x_{i} = \bigcap_{x \in \Pi_{n+1} \alpha} B^x_{i}$.

For $n < i < \omega$, let $A^x_{i} = A_i \cap \{ x \mid \nu \in x \to (\pi_{\beta_i}^x, \pi_{\delta_i}^x (\nu) \in x, \pi_{\beta_i}, \pi_{\delta_i} (\nu) \in x) \}$.

For $i \leq n$, let $f^x_{\beta_i} (y)$ be obtained from $f^x_{\beta_i} (y)$, restricted to $B^x_{i}$'s.

For $x \in A^x_{m}, m > n, \text{ and } \langle y, y \rangle \text{ in } \prod_{n<i \leq m} A^x_{i} \times B^x_{i}$ with $x < y$, let

$$f^m_n(x)(y_{n+1},...,y_{m},\nu_{n+1},...\nu_{m}) = f^m_n(x)(\nu_{n+1}(y_{n+1})...\nu_{m}(y_{m})).$$

Then $p$ is as desired.
Using a similar, and actually simpler argument, we get:

**Lemma 3.4.** Both \((\mathbb{P}_0, \leq^*)\) and \((\mathbb{P}, \leq)\) have the \(\mu^+\)\(\text{-}\)c.c.

**Lemma 3.5.** Let \(n > 0\). \((\mathbb{P}_0, \leq^*)\) preserves cardinals in the interval \([\kappa_n^{++}, \kappa_{n+1}]\).

**Proof.** Suppose otherwise. Let \(n\) be such that some regular \(V\)-cardinal \(\tau \in [\kappa_n^{++}, \kappa_{n+1}]\) is collapsed. Let \(p \in \mathbb{P}_0\), and \(\lambda < \tau\) be such that \(p \Vdash \tilde{h} : \lambda \to \tau\) is onto. Fix \(\alpha < \lambda\). We will define \(\theta \leq \kappa_n^{++}\) and \(\langle p_\eta, \alpha_\eta \mid \eta < \theta \rangle\) by induction of \(\eta\), such that:

1. \(p_\eta \in \mathbb{P}_0, p_\eta \leq^* p, \alpha_\eta \in \tau\),
2. \(\langle p_\eta \restriction [n+1, \omega) \mid \eta < \theta \rangle\) is \(\leq^*\)\(\text{-}\)decreasing,
3. \(p_\eta \Vdash_\mathbb{P}_0 \tilde{h}(\alpha) = \alpha_\eta\).

Let \(\alpha_0\) and \(p_0 \leq^* p\) be such that \(p_0 \Vdash_\mathbb{P}_0 \tilde{h}(\alpha) = \alpha_0\). Suppose we have defined \(p_\xi, \alpha_\xi\), for all \(\xi < \eta\). If \(\eta = \kappa_n^{++}\), set \(\theta = \eta\) and stop. Otherwise let \(q \leq^* p\) be such that \(q \restriction n + 1 = p \restriction n + 1\) and \(q \restriction [n+1, \omega) \leq^* p_\xi \restriction [n+1, \omega)\) for all \(\xi < \eta\). We can find such a condition because \((\mathbb{P}_0 \restriction [n+1, \omega), \leq^*\) is \(\kappa_{n+1}\)\(-\)closed.

Suppose that there is \(r \in \mathbb{P}_0, r \leq^* q\) and \(\beta \notin \{\alpha_\xi \mid \xi < \eta\}, \) such that \(r \Vdash_\mathbb{P}_0 \tilde{h}(\alpha) = \beta\). Then let \(\alpha_\eta = \beta\) and \(p_\eta = r\). Otherwise, set \(\theta = \eta, q_\alpha := q\), and stop.

**Claim 3.6.** \(\theta < \kappa_n^{++}\).

**Proof.** Otherwise \(\langle \pi_{n+1}(p_\eta) \mid \eta < \kappa_n^{++}\rangle\) is an antichain in \(\mathbb{P}_{0n+1}\) of size \(\kappa_n^{++}\). Contradiction with Proposition 3.3.

It follows that each \(q_\alpha\) is defined. Note that for all \(\alpha, q_\alpha \restriction n + 1 = p \restriction n + 1\). Let \(X_\alpha = \{\alpha_\eta \mid \eta < \theta\}\). Then \(q_\alpha \Vdash_\mathbb{P}_0 \tilde{h}(\alpha) \in X_\alpha\). Doing this inductively on \(\alpha < \lambda\), we arrange that \(\langle q_\alpha \restriction [n+1, \omega) \mid \alpha < \kappa\rangle\) is \(\leq^*\)\(\text{-}\)decreasing. Finally let \(X = \bigcup_{\alpha < \lambda} X_\alpha\), and let \(q \leq^* p\) be such that for all \(\alpha < \lambda, q \restriction [n+1, \omega) \leq^* q_\alpha \restriction [n+1, \omega)\) and \(q \restriction n + 1 = p \restriction n + 1\). Then \(q \Vdash_\mathbb{P}_0 \text{ran}(\tilde{h}) \subseteq X\), but \(|X| < \tau\). Contradiction.

**Corollary 3.7.** \(\mathbb{P}_0\) preserves \(\mu\).

For conditions \(p, q \in \mathbb{P}\), we say that \(p\) and \(q\) are *tail equivalent*, if for some large enough \(n, p \restriction [n, \omega) \sim q \restriction [n, \omega)\), as defined earlier, restricted to \(\mathbb{P} \restriction [n, \omega)\). In this case we write \(p \sim_t q\). Denote the tail-equivalence class of \(p\), by \(t(p) := \{q \mid p \sim_t q\}\).

**Definition 3.8.** Let \(\mathbb{D} := \{t(p) \mid p \in \mathbb{P}\} \) with the ordering \(t(p) \leq_{\mathbb{D}} t(q)\) if for some \(n, p \restriction [n, \omega) \leq^* q \restriction [n, \omega)\).

By considering the map \(p \mapsto t(p)\), we get the following:

**Proposition 3.9.** Both \(\mathbb{P}\) and \(\mathbb{P}_0\) project to \(\mathbb{D}\).

**Proposition 3.10.** Suppose that \(H\) is \(\mathbb{D}\)-generic, \(G_0\) is \(\mathbb{P}_0/H\)-generic, and \(p \in \mathbb{P}/H\). Then there is some \(n\), such that \(1^\perp p \restriction [n, \omega) \in G_0\).

**Lemma 3.11.** Let \(H\) be \(\mathbb{D}\)-generic. \(\mathbb{P}/H\) has the \(\mu\)\(-\)c.c.

**Proof.** Suppose \(\{p_\eta \mid \eta < \mu\}\) are conditions in \(\mathbb{P}/H\). I.e. for each \(\eta, t(p_\eta) \in H\). By passing to an unbounded subset of \(\mu\), we may assume that there is \(\bar{n} < \omega\), and \(\bar{x}\) of length \(\bar{n}\), such that all conditions have length \(\bar{n}\) and Prkry stem \(\bar{x}\). Let \(G_0\) be \(\mathbb{P}_0/H\)-generic. Then for all \(\eta\), there is some \(n_\eta > \bar{n}\), such that \(1^\perp p_\eta \restriction [n_\eta, \omega) \in G_0\).
Corollary 3.12. \( P \) preserves \( \mu \).

4. The Prikry Lemma

First we show the diagonal lemma:

Lemma 4.1. Suppose that \( p \) is a condition of length \( l \) and for all \( (x, \nu) \in A^p_1 \times B^p_1 \) with \( \nu \in x \), we have \( p_{x,\nu} \leq * p \triangleleft (x, \nu) \). Suppose also that:

1. There are \( \langle \beta_n \mid l < n < \omega \rangle \), such that every \( \beta_n \) is \( E_\nu \beta_n \) and for all \( y \in A^l_1 \), for all \( h \), with \( y < h \),
   \[
   (f^{p_{x,\nu}}(y)(\pi^{x,\nu}(h))) \upharpoonright \text{dom}(f^{p_{x,\nu}}(y)(\pi^{x,\nu}(h))) \setminus a^p_1(y) | \nu \in x, x < y
   \]
   are pairwise compatible, where \( \pi^{x,\nu} \) is the projection from the \( \beta_n \)’s to the \( \beta_n \)’s.

2. \( (p_{x,\nu} \mid \nu + 1, \omega) \) are \( \preceq \)-pairwise compatible.

Then there is a direct extension \( q \leq * p \), such that if \( r \) is a nondirect extension of \( q \), then for some \( x, \nu \), we have that \( r \leq p_{x,\nu} \). Moreover, we can choose \( q \) so that for all \( x \in A^l_1 \), \( a^q_1(x) = a^p_1(x) \).

Proof. For simplicity assume that \( lh(p) = 1 \). Denote \( p_{x,\nu} = \langle x_0, f_0^{x,\nu}, x, f_1^{x,\nu}, A_2^{x,\nu}, F_2^{x,\nu}, \ldots \rangle \), and for \( n > 1 \), \( F_n^{x,\nu}(y) = \langle a_n^{x,\nu}(y), A_n^{x,\nu}(y), f_n^{x,\nu}(y) \rangle \). By taking diagonal intersections, by item (2), we can assume that for all \( n > 1 \), for all \( x, \delta, \eta \), for all \( y \in A^0_n \) with \( x < y, \delta < \nu \) and for all \( h \) with \( y < h \), \( \langle a_n^{x,\nu}(y), A_n^{x,\nu}(y), f_n^{x,\nu}(y)(\pi_1(h)) \rangle \) and \( \langle a_n^{x,\nu}(y), A_n^{x,\nu}(y), f_n^{x,\nu}(y)(\pi_2(h)) \rangle \) are pairwise compatible, where \( \pi_1 \) and \( \pi_2 \) project to the maximal coordinates of \( p^{x,\nu} \) and \( p^{x,\nu} \), respectively, from some coordinate above both.

For every \( \nu \), we have that \( B_\nu := \{ x \in A^l_1 \mid \nu_\nu \in A^p_1(x) \} \subseteq U_1 \). Set \( A^l_1 = \Delta_\nu B_\nu \). For \( y \in A^l_1 \), set \( A^l_1(y) = a^l_1(y), A^l_1(y) = A^l_1(y) \). For \( n > 1 \), let \( A^l_n = \Delta A^l_n := \{ z \mid z \in \bigcap_{x \in A^l_1} A^l_n \}. \) For \( n > 1 \) and \( y \in A^l_n \), set:

1. \( A^l_n(y) = \bigcup_{x \in A^l_1, \nu \in x} a^l_n(x, \nu)(y), \) and
2. \( A^l_n(y) = \bigcap_{x \in A^l_1, \nu \in x} \pi^{x,\nu}(y)(\pi_1(h)), \) are pairwise compatible.

This is possible since there is a maximal element for the \( a \)’s unboundedly often. And by choosing the \( a \)'s inductively for \( n \), we maintain the last item of 2.4. Then, for \( n > 1 \), let \( A^0_n = A^l_n \cap \{ x \cap \nu \in x \rightarrow \pi^{\beta_n,\nu}(\nu) \in x \}. \) For every \( (x, \nu) \) and \( h \in \prod_{n \in \omega} A^l_n \times B^l_n \), let \( \pi_{x,\nu}(h) \) be the corresponding pointwise projection of \( h \) from the maximal coordinates of \( p_{x,\nu} \) to \( p \). Let \( \pi_{q,\nu}(h) \) be the projection from the maximal coordinates of \( q \) to \( p_{x,\nu} \), and let \( \pi_{q,p}(h) \) be the projection from the maximal coordinates of \( q \) to \( p \).
Then set
\[ q \]

there is a condition
\[ D \]

Proof. assumption let
\[ p \]

the diagonal lemma. Apply the diagonal lemma to the conditions
\[ \text{such that} \]

Remark 1 follows that
\[ x \]

the Diagonal lemma, when diagonalizing over the
\[ p \]

for all
\[ l < k < n \]

That is because when running the argument above, by induction, we may assume that
\[ x,\nu \]

Lemma 4.3. (Prikry lemma) Suppose that
\[ D \]

is an open dense set and
\[ n > \text{lh}(p) \]

Then there is a condition
\[ q \leq^* p \]

such that for all
\[ r \leq q \]

with length
\[ n \]

and
\[ r \]

in
\[ D \]

, then
\[ r \]

is in
\[ D \]

. 

Proof. By induction on
\[ n-l \]

. If
\[ n = \text{lh}(p) + 1 \]

, the result follows from the Diagonal lemma. Suppose
\[ n > \text{lh}(p) + 1 \]

. For every
\[ \langle x,\nu \rangle \]

, such that
\[ q^\frown \langle x,\nu \rangle \]

is defined, by the inductive assumption let
\[ p_{x,\nu} \leq^* q^\frown \langle x,\nu \rangle \]

be such that for all
\[ r \leq p_{x,\nu} \]

with length
\[ n \]

, if there is
\[ r' \leq^* r \]

in
\[ D \]

, then
\[ r \]

is in
\[ D \]

. 

Defining these condition inductively, we arrange that they satisfy the assumptions of the diagonal lemma. Apply the diagonal lemma to the conditions
\[ p_{x,\nu} \]

and
\[ p \]

to get
\[ q \leq^* p \]

, such that
\[ q^\frown \langle x,\nu \rangle \leq^* p_{x,\nu} \]

for all
\[ x,\nu \]

. Then
\[ q \]

is as desired. For if
\[ r \leq q \]

is with length
\[ n \]

, let
\[ x,\nu \]

be such that
\[ r \leq p_{x,\nu} \]

. Now, if
\[ r' \leq^* r \]

is in
\[ D \]

, then by the way we chose
\[ p_{x,\nu} \]

, it follows that
\[ r \]

is in
\[ D \]

. 

Remark 1. We can define
\[ q \]

as above so that for all
\[ l \leq k < n \]

and
\[ x \in A_k \]

, 
\[ a_k^q(x) = a_k^q(x) \]

. That is because when running the argument above, by induction, we may assume that for all
\[ l \leq k < n \]

, for all
\[ x,\nu \]

and
\[ y \in A_k \]

, 
\[ a_k^p(x,\nu) = a_k^p(y) \]

. Then, as in the proof of the Diagonal lemma, when diagonalizing over the
\[ p_{x,\nu} \]

’s we get that for all
\[ l \leq k < n \]

and
\[ x \in A_k \]

, 
\[ a_k^q(x) = a_k^q(x) \]

. 

Lemma 4.3. (Prikry lemma) Suppose that
\[ D \]

is an open dense set and
\[ p \]

is a condition with length
\[ l \]

. Then there is some
\[ n \]

and
\[ q \leq^* p \]

, such that for all
\[ \bar{x},\bar{\nu} \]

of length
\[ n \]

, such that
\[ q^\frown \langle \bar{x},\bar{\nu} \rangle \]

is defined, we have that
\[ q^\frown \langle \bar{x},\bar{\nu} \rangle \]

is in
\[ D \]

. 

Proof. First by shrinking measure one sets, we may assume that for some fixed
\[ n \]

, for all
\[ r \leq p \]

of length
\[ n+l \]

, there is some
\[ r' \leq^* r \]

such that
\[ r' \in D \]

. Let
\[ q \leq^* p \]

be given by the above corollary applied to
\[ D \]

. Then every
\[ n \]

-step extension of
\[ q \]

is in
\[ D \]

. 

Lemma 4.4. For every
\[ p \in \mathbb{P} \]

and formula
\[ \phi \]

, there is
\[ q \leq^* p \]

, such that
\[ q \]

decides
\[ \phi \]

. 

Proof. Apply the Prikry lemma for the set
\[ \{ q \mid q \parallel \phi \} \]

to find
\[ p' \leq p \]

and
\[ n \]

, such that every
\[ n \]

-step extension of
\[ p' \]

is in
\[ D' \]

. Then by shrinking measure one sets, in a rather standard way, we obtain
\[ q \leq^* p' \]

, such that all
\[ n \]

-step extensions of
\[ q \]

decide
\[ \phi \]

the same way. Then
\[ q \]

decides
\[ \phi \]

. 

Corollary 4.5. \( \mathbb{P} \) does not add bounded subsets of \( \kappa \)
Proof. This follows from the Prikry property and since \( \langle P, \leq^* \rangle \) is \( \kappa \)-closed. \( \Box \)

**Corollary 4.6.** \( P \) preserves cardinals up to and including \( \kappa \).

5. The generic extension

Prepare the ground model \( V \), such that the supercompactness of \( \kappa \) is preserved by forcing with \( P_0 \). Since \( P_0 \) is \( \kappa_0 \)-closed, and so does not add subsets of \( \kappa \), by starting with a model of GCH, we have that in \( V \), \( 2^\tau = \tau \) for all inaccessible \( \tau < \kappa \). Also, in \( V \), GCH\( \geq \kappa \) holds.

Let \( G \) be \( P \)-generic. Let \( \langle x_n \mid n < \omega \rangle \) be the diagonal supercompact Prikry sequence added by \( G \). Then \( \bigcup_n x_n = \kappa_\omega \) and \( V[G] \models (\forall i < \omega) \text{cf}(\kappa_i) = \omega \) and \( \mu = \kappa^+ \). Next we show that the forcing blows up the powerset of \( \kappa \).

**Lemma 5.1.** Suppose \( n < \omega, \alpha < \mu^+ \), and \( p \) is such that \( n \geq \text{lh}(p) \) and for all \( y \in A_n^p \), \( \alpha \in a^p(y) \). Then \( D_{n,\alpha} := \{ q \mid \text{lh}(q) > n, (\exists \beta : [x \mapsto \beta_x]_{U_n})(\forall h \in \text{dom}(f_n^p(x)))f_n^p(x)(h)(\alpha) = \beta_x \} \) is dense below \( p \).

*Proof.* Let \( q \leq p \) and \( \text{lh}(q) > n \). Say \( q \leq^* p \text{-} \langle \vec{x}, \vec{\nu} \rangle \), and let \( \nu \) is the \( n - \text{lh}(p) \) th element of the sequence \( \vec{\nu} \). Then let \( \beta := \pi_{[x \mapsto \max(a^p_n(x))]}_{U_n} (\nu) \). Denote \( \beta = [x \mapsto \beta_x]_{U_n} \). Then by definition of the \( Q \)-modules, we have that for \( U_n \)-almost all \( x \), for all \( h \in \text{dom}(f_n^p(x)) \), \( f_n^p(x)(h)(\alpha) = \beta_x = \pi_{\max(a^p_n(x), \alpha)}(\nu_x) \). \( \Box \)

For \( p \) in \( D_{n,\alpha} \), define \( g^p_n(\alpha) = \beta \), where \( \beta \) witnesses that \( p \) is in that set. Let 

\[
F := \bigcup_{p \in G, n \geq \text{lh}(p), y \in A_n^p} a^p_n(y).
\]

Note that by genericity of the Prikry sequence and definition of \( P \), this is the same as taking \( F = \bigcup_{p \in G, n \geq \text{lh}(p)} a^p_n(x_n) \). Define \( g^*_n : F \to \kappa_n \) by \( g^*_n(\alpha) = g^p_n(\alpha) \) for some \( p \) in \( G \cap D_{n,\alpha} \), if such exists, and \( 0 \) otherwise.

**Lemma 5.2.** \( F \) is unbounded in \( \mu^+ \)

*Proof.* Let \( \alpha < \mu^+ \). We claim that the set \( D_{\alpha} := \{ p \mid (\exists \alpha' > \alpha)(\exists i \geq \text{lh}(p)) (\forall y \in A_i^p) \alpha' \in a^p_i(y) \} \) is dense. Let \( p \) be given. Since:

\[
\beta_0 := \sup_{n \geq \text{lh}(p), y \in A_n^p, h \in \text{dom}(f_n^p(y))} \text{dom}(f_n^p(y)(h)) < \mu^+,
\]

we have that \( \beta := \max(\beta_0, \alpha) < \mu^+ \). Take \( \alpha' \) with \( \beta < \alpha' < \mu^+ \). Now we can extend \( p \) to a condition \( q \), so that for some \( n > \text{lh}(q) \), for all \( y \in A_n^q \), we have that \( \alpha' \in a^q_n(y) \). \( \Box \)

Remark 2. By a similar argument, we get that \( F \cap \mu \) is unbounded in \( \mu \).

**Lemma 5.3.** If \( \alpha < \beta \) are both in \( F \), then for all large \( n \), \( g^*_n(\alpha) < g^*_n(\beta) \).

*Proof.* Let \( p_1, p_2 \) in \( G \) witness that \( \alpha, \beta \in F \). We can find a common extension \( p \in G \), such that for all \( n \geq \text{lh}(p) \), for all \( y \in A_n^p \), \( \{\alpha, \beta\} \subset a_n^p(y) \). We will show that for all \( n \geq \text{lh}(p) \), \( g^*_n(\alpha) < g^*_n(\beta) \). To this end, let \( q \in G \) be such that \( q \leq p \) and \( \text{lh}(q) > n \). Let \( q \leq^* p \text{-} \langle \vec{x}, \vec{\nu} \rangle \), and let \( \nu \) is the \( n - \text{lh}(p) \) th element of the sequence \( \vec{\nu} \). Then let \( \delta := \pi_{[x \mapsto \max(a^p_n(x))]}_{U_n} (\nu) \) and \( \delta' := \pi_{[x \mapsto \max(a^p_n(x))]}_{U_n} (\beta)(\nu) \). Then by definition of the \( Q \)-modules, we have that for \( U_n \)-almost all \( x \), for all \( h \in \text{dom}(f_n^p(x)) \), \( f_n^p(x)(h)(\alpha) = \delta_x < \delta'_x = f_n^p(x)(h)(\beta) \). So, \( g^*_n(\alpha) = \delta < \delta' = g^*_n(\beta) \).
We have that every $g_n^*$ has range $\kappa_n$. Next we use the genericity of $(x_n \mid n < \omega)$ to define functions with ranges in $\kappa_n^0 := |x_n|$. Now, for all $\eta$, let $F_n^\eta$ be the function such that $[F_n^\eta]_{\mathcal{U}_n} = \eta$. In $V[G]$, define functions $\langle t_\alpha \mid \alpha < \mu^+ \rangle$ in $\prod_n \kappa_n^0$ by

$$t_\alpha(n) := F_n^\eta(x_n).$$

Then $\langle t_\alpha \mid \alpha \in F \rangle$ are increasing sequences in $\prod_n \kappa_n^0$ mod finite.

**Corollary 5.4.** $V[G] \models 2^\kappa = \mu^+$.

### 6. NO VERY GOOD SCALE

In this section we show that there is no very good scale at $\kappa$ in $V[G]$. Suppose for contradiction, that in $V[G]$, $\langle f_\alpha \mid \alpha < \mu \rangle$ is a very good scale in some product $\prod_n \tau_n$, of regular cardinals with supremum $\kappa$. For every $n$ there is some $n'$, such that $\tau_n < \kappa_{n'}$. Suppose for simplicity that $n' = n$. The general case is similar. Also suppose for simplicity that all of this is forced by the empty condition.

**Proposition 6.1.** For all $\alpha < \mu$ and $p \in \mathbb{P}_n$, there is $q \leq^* p$, such that every $n + 1$-step extension of $q$ decides a value of $f_\alpha(n)$, and such that for all $k \leq n$, $x \in A^q_k$, $a^q_k(x) = a^p_k(x)$.

**Proof.** Let $D := \{ q \mid \exists \gamma (q \forces f_\alpha(n) = \gamma) \}$; this is clearly a dense open set. Then by Corollary 4.2, we get $q \leq^* p$ such that for all $r \leq q$ with length $n + 1$, if there is $r' \leq^* r$ in $D$, then $r$ is in $D$.

**Claim 6.2.** For all $r \leq p$ with $\text{lh}(r) = n + 1$, there is $r' \leq^* r$ with $r' \in D$.

**Proof.** Fix such $r$; say $x := x_n^r$. Then $r \forces \dot{f}_\alpha(n) < \kappa_x$. Apply the Prikry property to $\langle \dot{f}_\alpha(n) = \gamma \rangle$, for all $\gamma < \kappa_x$, to construct a $\leq^*$-decreasing sequence $\langle r_\gamma \mid \gamma < \kappa_x \rangle$ of direct extensions of $r$, deciding these formulas. Then let $r'$ be stronger than each $r_\gamma$; $r' \in D$.

It follows that every $r \leq q$ with length $n + 1$ is in $D$. Also, by Remark 1, for all $k \leq n$, $x \in A^q_k$, $a^q_k(x) = a^p_k(x)$.

**Remark 3.** Since $(\mathbb{P}, \leq^*)$ is $\kappa_0$-closed, the above proposition also works for functions in $\prod_n \kappa_n^0$, $\prod_n \kappa_n^0$, $\prod_n (\kappa_n^0)^+$, etc. (recall $\kappa_0^0 = |x|$ for $x \in \mathcal{P}_k(\kappa_n)$)

Now let $H$ be $\mathbb{D}$-generic induced from $G$, and let $G_0$ be $\mathbb{P}_0/H$-generic over $V$. Since $\mathbb{P}/H$ has the $\mu$-chain condition there is a club subset of $\mu$, $E \in V[H]$, such that every point in $E$ is very good, and of course $E$ remains a club in $V[G_0]$.

For two functions $f, g$, we will write $f \leq_n g$ to denote that for all $k \geq n$, $f(k) < g(k)$.

**Lemma 6.3.** In $V[G_0]$, there is $n < \omega$, and a $\kappa$-club $C \subset \mu$, such that for all $\alpha < \beta$ in $C$, there is $p \in G_0$, such that $p \Vdash \dot{f}_\alpha <_n \dot{f}_\beta$.

**Proof.** For every $\delta < \mu$ with $\omega < \text{cf}^V(\delta) = \text{cf}^{V[G_0]}(\delta) < \kappa$, let $Y_\delta \in V$ be any club in $\delta$ of order type $\text{cf}^V(\delta)$. Enumerate $\mathcal{P}^V(Y_\delta)$ by $\{ C_{\delta,i} \mid i < 2^{\text{cf}(\delta)} \}$. Since $\kappa$ is strong limit, we have that $2^{\text{cf}(\delta)} < \kappa$. So, by applying the Prikry property, we can produce a condition $p_\delta$ of length 0, such that for each $i$, and $n < \omega$, $p_\delta$ decides whether $C_{\delta,i}$ and $n$ witness very goodness of $\delta$. By density, we choose each $p_\delta \in G_0$. By assumption, for club many $\delta$’s there is some $i, n$ such that $C_{\delta,i}$ and $n$ witness very goodness.
Let \( j : V[G_0] \to M \) be a \( \mu \)-supercompact embedding with critical point \( \kappa \). Set \( \rho := \sup j^\# \mu \). Then by elementarity, there is a condition \( p^* \in j(G_0) \), \( n < \omega \), and \( C^* \in M \) of order type \( \text{cf}^M(\rho) = \mu \), such that \( p^* \) forces that \( C^*, n \) witness that \( \rho \) is very good. Let \( C := \{ \gamma < \mu \mid j(\gamma) \in C^* \} \).

Then \( C \) is a club in \( \mu \). Now suppose that \( \alpha < \beta \) are in \( C \) and \( q \in G_0 \). Let \( r^* \leq^* j(q), p^* \) be in \( j(G_0) \). Then \( r^* \forces_{j(p)} j(\check{f})_{j(\alpha)} < n j(\check{f})_{j(\beta)} \) (since \( p^* \) forces it). So, by elementarity, there is a condition \( p \in G_0, p \leq^* q \), such that \( p \forces \check{f}_\alpha < n \check{f}_\beta \).

□

Let \( C \) be a \( \mathbb{P}_0 \) name for a club as above and suppose that the empty condition forces (over \( \mathbb{P}_0 \)) that \( C, n \) are as above. I.e. for all \( p \in \mathbb{P}_0 \), and \( \alpha < \beta < \mu \), if \( p \forces_{\mathbb{P}_0} \alpha, \beta \in \hat{C} \), then there is \( q \leq^* p \), such that \( q \forces \check{f}_\alpha < n \check{f}_\beta \).

**Lemma 6.4.** For all \( \tau < \kappa_\omega \) and \( p \in \mathbb{P}_0 \), there is \( X \subset \mu \) in \( V \) with \( |X| = \tau \) and \( r \leq^* p \), such that \( r \forces_{\mathbb{P}_0} X \subset \hat{C} \).

**Proof.** Let \( m \) be such that \( \tau < \kappa_m \). We use the following claim.

**Claim 6.5.** For all \( \alpha < \mu \), for all \( p \), there is \( \beta > \alpha \) and \( q \leq^* p \), such that \( \pi_m(q) = \pi_m(p) \) and \( q \forces_{\mathbb{P}_0} \beta \in \hat{C} \).

**Proof.** Construct \( \leq^* \)-decreasing sequence of conditions \( \langle g_k \mid k < \omega \rangle \) and an increasing sequence of points \( \langle \alpha_k \mid k < \omega \rangle \), such that \( \alpha_0 = \alpha \), every \( q_k \forces_{\mathbb{P}_0} \hat{C} \cap \langle \alpha_k, \alpha_{k+1} \rangle \neq \emptyset \), and \( \pi_m(q_k) = \pi_m(p) \). We can do this by standard arguments since \( \mathbb{P}_{0,m} \) has the \( \kappa_m^{++} \)-c.c. and \( \mathbb{P} \upharpoonright [m, \omega) \) is \( \kappa_m \)-closed. Then let \( \beta = \sup_k \alpha_k \) and let \( q \leq^* q_k \) for all \( k \). Then \( q \forces_{\mathbb{P}_0} \beta \in \hat{C} \).

\( \Box \)

Fix \( p \). We will construct a sequence \( \langle \beta_\eta \mid \eta < \tau \rangle \) and \( \langle q_\eta \mid \eta < \tau \rangle \), such that for each \( \eta \), \( \pi_m(q_\eta) = \pi_m(p) \) and \( \langle q_\eta \upharpoonright [m, \omega) \mid \eta < \tau \rangle \) is \( \leq^* \)-decreasing.

Suppose we have defined the sequences up to \( \eta \). Let \( q \leq^* p \) be such that \( \pi_m(q) = \pi_m(p) \) and \( q \upharpoonright [m, \omega) \leq^* q_\xi \upharpoonright [m, \omega) \) for all \( \xi < \eta \). Let \( q_\eta \leq^* q, \beta_\eta > \sup_\xi < \eta \beta_\xi \) be given by the claim applied to \( q \) and \( \sup_\xi \beta_\xi \).

Finally let \( r \leq^* p \) be such that \( \pi_m(r) = \pi_m(p) \) and \( r \upharpoonright [m, \omega) \leq^* q_\eta \upharpoonright [m, \omega) \) for all \( \eta < \tau \). Set \( X = \{ \beta_\eta \mid \eta < \tau \} \). Then \( r \forces_{\mathbb{P}_0} X \subset \hat{C} \).

\( \Box \)

Apply the above lemma to find a condition \( r \in \mathbb{P}_0 \) and \( X \subset \mu \) of size \( \kappa_m^+ \), such that \( r \forces_{\mathbb{P}_0} X \subset \hat{C} \). For every \( \alpha \in X \), let \( p_\alpha \leq^* r \) be given by Proposition 6.1. I.e. every \( q \leq p_\alpha \) with length \( n + 1 \) decides \( \check{f}_\alpha(n) \), and for all \( k \leq n, x \in A_k^q, a^\alpha_k(x) = a^\alpha_k(x) \). \( \mathbb{P} \upharpoonright [n + 1, \omega) \) is \( \kappa_n \)-closed and \( |X| = \kappa_m^+ \). So by defining the \( p_\alpha \)’s inductively, we arrange that \( \langle p_\alpha \upharpoonright [n + 1, \omega) \mid \alpha \in X \rangle \) is \( \leq^* \)-decreasing.

Consider \( \{ \pi_{n+1}(p_\alpha) \mid \alpha \in X \} \subset \mathbb{P}_{0,n+1} \). By the same \( \Delta \)-system argument as in Proposition 3.3, there is an unbounded \( X' \subset X \), such that \( \{ \pi_{n+1}(p_\alpha) \mid \alpha \in X' \} \) are pairwise compatible. But that means \( \{ p_\alpha \mid \alpha \in X' \} \) are pairwise compatible with respect to \( \leq^* \).

For all \( \alpha, \beta \) in \( X' \), let \( p_{\alpha, \beta} \leq^* p_\alpha, p_\beta \) be such that \( p_{\alpha, \beta} \forces_{\mathbb{P}_0} \check{f}_\alpha < n \check{f}_\beta \). Let \( r_{\alpha, \beta} \leq p_{\alpha, \beta} \) be of length \( n + 1 \) and of the form \( r_{\alpha, \beta} = p_{\alpha, \beta}(\check{x}, \check{u}) \), for some \( \check{x}, \check{u} \). But then since for all \( k \leq n, x \in A_k^q, a^\alpha_k(x) = a^\alpha_k(x) \), we have that there are \( \check{x}_{\alpha, \beta}, \check{u}_{\alpha, \beta} \), such that:

- \( r_{\alpha, \beta} \leq^* p_\alpha(\check{x}_{\alpha, \beta}, \check{u}_{\alpha, \beta}) \)
- \( r_{\alpha, \beta} \leq^* p_\beta(\check{x}_{\alpha, \beta}, \check{u}_{\alpha, \beta}) \)
Denote $h_{\alpha,\beta} := \langle \vec{x}_{\alpha,\beta}, \vec{v}_{\alpha,\beta} \rangle$. The number of possible $h_{\alpha,\beta}$'s is $\kappa_n$, and $|X'| = \kappa_n^{++} = (2^{\kappa_n})^+$. By Erdos-Rado, the function $\langle \alpha, \beta \rangle \mapsto h_{\alpha,\beta}$ has a homogenous set $Y$ is size $\kappa_n^+$. Let $\langle \vec{x}, \vec{v} \rangle = h_{\alpha,\beta}$ for all $\alpha, \beta$ in $Y$.

For all $\alpha \in Y$, let $\gamma_\alpha < \kappa$ be such that, $p_\alpha \Vdash \langle \vec{x}, \vec{v} \rangle \models \hat{f}_\alpha(n) = \gamma_\alpha$. (Here we use that $p_\alpha$ is as in the conclusion of Proposition 6.1.) Suppose that $\alpha < \beta$ are both in $Y$. Since $r_{\alpha,\beta} \leq p_{\alpha,\beta}$ and $p_{\alpha,\beta} \Vdash \hat{f}_\alpha \leq \hat{f}_\beta$, we have that $r_{\alpha,\beta} \models \hat{f}_\alpha(n) < \hat{f}_\beta(n)$. But $r_{\alpha,\beta} \leq^* p_\alpha \Vdash \langle \vec{x}, \vec{v} \rangle, p_\beta \Vdash \langle \vec{x}, \vec{v} \rangle$, so $\gamma_\alpha < \gamma_\beta$.

But then $\{ \gamma_\alpha | \alpha \in Y \}$ is a subset of $\kappa$ of size $\kappa_n^+$. Contradiction.

7. Bad scale

Recall that we prepared the ground model $V$, so that the supercompactness of $\kappa$ is preserved by forcing with $\mathbb{P}_0$. In $V$, fix a scale $\langle g^*_\alpha | \gamma < \mu \rangle \in V$ in $\prod \kappa_\alpha^+$. Set $S := \{ \gamma < \mu | \omega < \text{cf}(\gamma) < \kappa, \gamma \text{ is a bad point for } \langle g^*_\alpha \mid \gamma < \mu \rangle \}$. By standard reflection arguments $S$ is stationary in $V$. Also, since $\mathbb{P}_0$ preserves $\mu$ and is $\kappa^+$-closed, $\langle g^*_\alpha \mid \gamma < \mu \rangle$ remains a bad scale after forcing with $\mathbb{P}_0$. More precisely, if $G_0$ is $\mathbb{P}_0$-generic, a point of cofinality less than $\kappa$ is bad in $V$ if it is bad in $V[G_0]$, and the set $S$ is stationary in $V[G_0]$ (since $\kappa$ remains supercompact in $V[G_0]$).

So if $H$ is $\mathbb{D}$-generic, since $\mathbb{P}_0$ projects to $\mathbb{D}$, we have that $S$ is stationary in $V[H]$. Then by the $\mu$-chain condition of $\mathbb{P}/\mathbb{D}$, $S$ is stationary after forcing with $\mathbb{P}$.

The next lemma will be used to show that a witness of goodness in the generic extension gives rise to a witness of goodness in the ground model. In particular, if a point is bad in $V$, then it is bad in $V[G]$.

**Lemma 7.1.** Let $\tau < \kappa$ be a regular uncountable cardinal in $V$ (and so in $V[G]$), and suppose $V[G] = A \subset ON, o.t.(A) = \tau$. Then there is a $B \in V$ such that $B$ is an unbounded subset of $A$.

**Proof.** Let $p \in G, p \Vdash \hat{h} : \tau \rightarrow \dot{A}$. By the Prikry lemma, define a $\leq^*$-decreasing sequence $\langle p_\alpha \mid \alpha < \tau \rangle$, such for every $\alpha < \tau, p_\alpha \leq^* p$ and there is $n_\alpha < \omega$, such that every $q \leq p_\alpha$ with length $n_\alpha$ decides $\hat{h}(\alpha)$. Then there is an unbounded $I \subset \tau$ and $n < \omega$ such that for all $\alpha \in I, n = n_\alpha$. Let $p'$ be stronger than all $p_\alpha$ for $\alpha < \tau$. By appealing to density, we may assume that $p' \in G$. Let $q \leq p$ be a condition in $G$ with length $n$, and set $B = \{ \gamma \mid (\exists \alpha \in I) q \Vdash \hat{h}(\alpha) = \gamma \}$. Then $B$ is as desired.

\[\square\]

Note that the above lemma already implies that the approachability property fails in $V[G]$, and so weak square also fails.

Recall that for every $x \in \mathcal{P}_\kappa(\kappa_n)$, $\kappa^n_x$ denotes $|x|$, which is a cardinal on a $U_n$-measure one set. Also, $\forall n < \omega, \forall \eta < \kappa_n^+$, we fixed $F^n_\eta : \mathcal{P}_\kappa(\kappa_n) \rightarrow V$, such that $[F^n_\eta]_{U_n} = \eta$. We may assume that $\forall x F^n_\eta(x) < (\kappa^n_x)^+$. Define in $V[G]$, $\langle g_\beta \mid \beta < \mu \rangle$ in $\prod \kappa_\alpha^+$ by:

$$g_\beta(n) = F^n_{\beta}(n)(x_n)$$

To show that this is a scale we need the following bounding lemma.

**Lemma 7.2.** Suppose that in $V[G]$, $h \in \prod \kappa_\alpha^+$. Then there is a sequence of functions $\langle H_n \mid n < \omega \rangle$ in $V$, such that $\text{dom}(H_n) = \mathcal{P}_\kappa(\kappa_n), H_n(x) < (\kappa^n_x)^+$ for all $x$, and for all large $n$, $h(n) \leq H_n(x_n)$.
Proof. Let $p$ force that $\dot{h} \in \prod_{n}(\kappa_{a}^{n})^{+}$. For simplicity, say $\text{lh}(p) = 0$.

Fix $n < \omega$. Let $p_{n} \leq^{*} p$ be such that every $n + 1$-step extension decides $\dot{h}(n)$. Let $q \leq^{*} p_{n}$, for all $n$. For all $\bar{z}, \bar{\nu}$ of length $n + 1$, such that $q \forces \dot{h}(n) = \gamma_{\bar{z}, \bar{\nu}}$, for $x \in A_{n}^{a}, \nu \in B_{n}^{a}$ with $\nu \in x$, define $H_{n}(x, \nu) = \sup \{ \gamma_{\bar{z}, \bar{\nu}} | z_{n} = x, \nu_{n} = \nu \} < \kappa_{a}^{n}$, where $z_{n}$ and $\nu_{n}$ denote the last elements of $\bar{z}$ and $\bar{\nu}$ respectively. Let $H_{n}(x) = \sup_{\nu \in B_{n}^{a}, \nu \in x} H_{n}(x, \nu) < (\kappa_{a}^{n})^{+}$.

Then $q$ forces that $\langle H_{n} | n < \omega \rangle$ is as desired.

□

Corollary 7.3. $\langle g_{\beta} | \beta < \mu \rangle$ is a bad scale in $V[G]$

Proof. $\langle g_{\beta} | \beta < \mu \rangle$ is a scale by the way we defined it and Lemma 7.2, (see for example the arguments in [1]). Also, by Lemma 7.1, if $\gamma$ is a good point in $V[G]$ for $\langle g_{\beta} | \beta < \mu \rangle$ with cofinality $\tau$ with $\omega < \tau < \kappa$, then $\gamma$ is a good point in $V$ for $\langle g_{\beta}^{a} | \beta < \mu \rangle$. Finally, the set of bad points $S$ is still stationary in $V[G]$.

□

We conclude with some questions.

Question 1. What can be said about the tree property at $\kappa$ in the above construction?

Question 2. Can we use short extenders and collapses to obtain the present construction for $\kappa = \aleph_{\omega}$?

References