The BMM Global-Local bijection for GL(n,q)

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Banff, March 2014
Let $G_n = \text{GL}(n, q)$, $\ell$ a prime not dividing $q$, $e$ the order of $q \mod \ell$. Unipotent characters of $G_n$ are constituents of $\text{Ind}_{B}^{G_n}(1)$ ($B$ a Borel) and are indexed by partitions of $n$. Denoted by $\chi_{\lambda}$, $\lambda$ a partition of $n$.

**Theorem (Fong-Srinivasan)** $\chi_{\lambda}$, $\chi_{\mu}$ are in the same $\ell$-block if and only if $\lambda$, $\mu$ have the same $e$-core.
Let $G_n = GL(n, q)$, $\ell$ a prime not dividing $q$, $e$ the order of $q$ mod $\ell$. Unipotent characters of $G_n$ are constituents of $\text{Ind}_{B}^{G_n}(1)$ ($B$ a Borel) and are indexed by partitions of $n$. Denoted by $\chi_{\lambda}$, $\lambda$ a partition of $n$.

Theorem (Fong-Srinivasan) $\chi_{\lambda}$, $\chi_{\mu}$ are in the same $\ell$-block if and only if $\lambda$, $\mu$ have the same $e$-core.
Alternatively: Unipotent blocks classified by pairs \((\lambda, k)\) (e-core, weight)

If \(B \leftrightarrow (\lambda, k)\), then \(\chi_\mu \in B\) iff \(\chi_\mu\) is a constituent of \(< R_L^G(\chi_\lambda)\), where \((L, \chi_\lambda)\) is an e-cuspidal pair

\(L\) (e-split Levi) is isomorphic to a product of \(k\) copies of tori of order \(q^e - 1\) and \(G_m\), \(G_m\) has e-cuspidal \(\chi_\lambda\).

\(N(L)/L\) isomorphic to \(W(L, \lambda) = Z_e \wr S_k = G(e, 1, k)\)
Broué, Malle, Michel: Global to Local Bijection for $G_n$:
Isometry $I_L^G$ maps $\phi_{\mu^*}$, character of $W_G(L, \lambda)$, to $\chi_\mu$, constituent of $R_L^G(\lambda)$ (up to sign), where $\mu^*$ is $e$-quotient of $\mu$. 
Similarly, have Isometry $I^M_L$, $M = e$-split Levi subgroup containing $L$, can choose $M = G_m \times GL(k, q^e)$.

Then: $R^G_MI^M_L = I^G_L \text{Ind}_{W_M(L, \lambda)}^{W_G(L, \lambda)}$. 
Let $L = G_n \times GL(k, q^e)$, an $e$-split Levi subgroup of $G_{n+k}$. If $\mu \vdash k$, define the Lusztig functor $L_\mu$ on $[A]$ where $A = \bigoplus_{n \geq 0} A_n$, $A_n$ the category of unipotent representations of $G_n$.

$L_\mu(\chi_\lambda) = R_{L}^{G_{n+k}}(\chi_\lambda \times \chi_\mu)$ where $L = G_n \times GL(k, q^e)$, and $\lambda, \mu$ are partitions of $n, k$ respectively.
Let $L = G_n \times \text{GL}(k, q^e)$, an $e$-split Levi subgroup of $G_{n+k}$. If $\mu \vdash k$, define the Lusztig functor $\mathcal{L}_\mu$ on $[\mathcal{A}]$ where $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$, $\mathcal{A}_n$ the category of unipotent representations of $G_n$.

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where $L = G_n \times \text{GL}(k, q^e)$, and $\lambda, \mu$ are partitions of $n, k$ respectively.
Let $\Gamma_n = Z_e \lhd S_n$, complex reflection group. 
$\text{Rep}(\Gamma_n) =$ Category of representations of $\Gamma_n$ over $\mathbb{C}$ and 
$\text{Rep}(\Gamma) = \bigoplus_n \text{Rep}(\Gamma_n)$.

Then $[\text{Rep}(\Gamma)]$ has basis indexed by e-tuples of partitions.

Parabolic subgroup $\Gamma_{n,k}$ of $\Gamma_{n+k}$ is of the form $\Gamma_n \times S_k$ where $S_k$ is a symmetric group.

Have Induction $\text{Rep}(\Gamma_{n,k}) \rightarrow \text{Rep}(\Gamma_{n+k})$. 
Reference: Shan-Vasserot, p.1010; Uglov 4.1, 4.2

Fock space $\mathcal{F}$ a vector space over $\mathbb{C}$ with standard basis $\mathcal{B}_1 = \{ | \lambda \rangle \}$ indexed by all partitions of $n \geq 0$.

There is also a $\mathbb{C}$-basis $\mathcal{B}_2 = \{ (\lambda_e, s_e) \}$ where $\lambda_e$ runs over $e$-tuples of partitions, $s_e$ is an $e$-tuple of integers summing up to 0.

Remark: Regard $\lambda_e$ as $e$-quotient, $s_e$ as a label for an $e$-core of $\lambda$. Both can be obtained from the Young diagram of $\lambda$. 
\( \mathcal{A}_n \) = category of unipotent representations of \( G_n \).

If \( \mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n \), [\( \mathcal{A} \)] (complexified Grothendieck group) is isomorphic to \( \mathcal{F} \) as a \( \mathbb{C} \)-vector space, since [\( \mathcal{A} \)] also has a basis indexed by partitions.

[Rep(\( \Gamma \))] isomorphic to \( \mathcal{F}^{(s)} \), subspace of \( \mathcal{F} \) with basis \( (\lambda_e, s) \) for fixed \( s \). Both have bases running over \( e \)-tuples of partitions.
Heisenberg Lie algebra $\mathfrak{h}$, generators $\langle B_k | k \in \mathbb{Z} - \{0\} \rangle$

with relations $[B_k, B_\ell] = k \frac{1-q^{-2nk}}{1-q^{-2k}} \delta_{k,-\ell}$
Leclerc-Thibon: Commuting operators $V_k$ ($k \geq 1$) in $\mathfrak{h}$ acting on $\mathcal{F}$.

$$V_k(|\lambda>) = \sum_{|\mu>} (-1)^{-s(\mu/\lambda)}|\mu>$$

where the sum is over all $\mu$ such that $\mu$ is obtained from $\lambda$ by adding $k$ e-skew hooks, such that the tail of each skew hook is not upon the head of another skew hook.

$s$ is the leg length of the skew hook.
More generally, we have $V_\rho \in U(\mathfrak{h})$ where $\rho$ is a composition:

If $\rho = \{\rho_1, \rho_2, \ldots\}$ then $V_\rho = V_{\rho_1} V_{\rho_2} \ldots$.

Then $S_\mu = \sum_\rho k_{\mu \rho} V_\rho$, operator in $U(\mathfrak{h})$, $k_{\mu \rho}$ are inverse Kostka polynomials.
$U(\mathfrak{h})$ acts on $[A] \leftrightarrow \mathcal{F}$ by $S_\mu$ (basis $\mathcal{B}_1$ of partitions, indexing unipotent characters).

Also, $U(\mathfrak{h})$ acts on $\mathcal{F}^{(s)} \leftrightarrow [\text{Rep}(\Gamma)]$ by $S_\mu$, (now on basis $\mathcal{B}_2$ of e-tuples of partitions, indexing $\text{Rep}(\Gamma)$.}
Main theorems:

Theorem. \( S_\mu \), acting on \( \mathcal{F} \), can be identified with Lusztig induction \( \mathcal{L}_\mu \) on \( [\mathcal{A}] \).

Theorem (Shan-Vasserot) Action of \( S_\mu \) on \( \mathcal{F}^{(s)} \) is identified with ordinary induction in \( [\text{Rep}(\Gamma)] \).
Theorem. $S_\mu$, acting on $\mathcal{F}$, can be identified with Lusztig induction $\mathcal{L}_\mu$ on $[A]$.

Two applications:

1. Interpretation of BMM bijection
2. Connection between some Brauer characters and Lusztig induction
(1) Interpretation of BMM Bijection:
Consider the map $\lambda \rightarrow (\lambda^*, s)$ where $\lambda^*$ is the $e$-quotient of $\lambda$ and $s$ labels the $e$-core of $\lambda$, between the basis $B_1$ of all partitions $|\lambda>$ and the basis $B_2$ of $(\lambda_e, s_e)$ where $\lambda_e$ are $\ell$-tuples of partitions.
Fix $k, \mu \vdash k$. The action of $S_\mu \in U(h)$ on $B_1$, interpreted as on $[\mathcal{A}]$, corresponds to Lusztig induction on the groups $G_n$. On the other hand, the action on $B_2$, interpreted as on $[\text{Rep}\Gamma]$, corresponds to ordinary induction on complex reflection groups.
Work done on blocks and decomposition matrices of finite reductive groups: Dipper-James, Geck, Gruber, Hiss, Kessar, Malle .. e.g. modular Harish-Chandra theory.

In Dipper-James theory, have $q$-Schur algebra $S_n$. 


$K, \mathcal{O}, k, \ell$-modular system

Dipper-James theory: $e$ is the order of $q \mod \ell$. Here $q \in k$, characteristic $\ell$.

The decomposition matrix of $S_n$ is square, has entries the multiplicities of irreducibles in Weyl modules.

There is a square part of the decomposition matrix of $G_n$, rows indexed by unipotent characters, columns by Brauer characters.

These two matrices are the same!
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(2) Back to space $\mathcal{F} \leftrightarrow [A]$ standard basis $\chi_\lambda$ (unipotent characters). Two canonical bases (Leclerc-Thibon, Uglov), analogous to Lusztig’s canonical bases.

\[ G^+(\lambda) = \sum d_{\lambda \mu} \chi_\mu \]
\[ G^-(\lambda) = \sum e_{\lambda \mu} \chi_\mu \]
By the work of Varagnolo-Vasserot on decomposition matrix of $S_n$, for large $\ell$ we have:

If $\lambda, \mu \vdash n$, $D = (d_{\lambda\mu})$ is the unipotent part of the decomposition matrix of $G_n$.

If $E = (e_{\lambda\mu})$, $E$ is the inverse transpose of $D$. 
The columns of $D$ express the unipotent characters of $G_n$ in terms of Brauer characters. Thus, the rows of $E$ express the Brauer characters of $G_n$ in terms of unipotent characters. Describe $G^{-}(\lambda) = \sum e_{\lambda\mu} \chi_{\mu}$. 
Example of the inverse decomposition matrix $E$ for $n = 4$, $e = 2$:

$$
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 & 0 \\
31 & -1 & 1 & 0 & 0 & 0 \\
22 & 1 & -1 & 1 & 0 & 0 \\
211 & -1 & 0 & -1 & 1 & 0 \\
1111 & 0 & 0 & 1 & -1 & 1 \\
\end{pmatrix}
$$

$G^{-}(211) = -\chi_4 - \chi_{22} + \chi_{211}$,

$G^{-}(22), G^{-}(211), G^{-}(1111)$ are Brauer characters.
Algorithm exists to compute these decomposition numbers in principle.

We wish to describe some of them by Lusztig induction.
Theorem. Let $\lambda \vdash n$, $\lambda = \mu + e\alpha$, $\mu \vdash m$, $\alpha \vdash k$, and let $\mu'$ be $e$-regular. Then the Brauer character represented by $G^-(\lambda)$ is equal to the Lusztig generalized character $R_{L}^{G_{n}}(G^{-}(\mu) \times \chi_{\alpha})$, where $n=m+ke$, $L = G_{m} \times GL(k, q^{e})$.

Proof. Leclerc-Thibon have proved that $G^{-}(\lambda) = S_{\alpha} G^{-}(\mu)$, so the proof follows from $S_{\alpha} = \mathcal{L}_{\alpha}$. 
An example of a decomposition matrix $D$ for $n = 4, \, e = 4$:

\[
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 \\
31 & 1 & 1 & 0 & 0 \\
211 & 0 & 1 & 1 & 0 \\
1111 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]
An example of the inverse of a decomposition matrix $D$ for $n = 6$, $e = 2$:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1
\end{pmatrix}
$$

Here the rows are indexed as: 6, 51, 42, 41^2, 3^2, 31^3, 2^3, 2^21^2, 21^4, 1^6

Source: GAP, MAPLE
In the above matrix:

The rows indexed by $1^6, 2^21^2, 3^2, 21^4, 41^2$ have interpretations in terms of $R_L^{G_n}$, with $L$ e-split Levi of the form $GL(3, q^2)$, $GL(2, q) \times GL(2, q^2)$, $GL(4, q) \times GL(1, q^2)$, as Brauer characters.

Row indexed by $3^2$: $L = GL(3, q^2): R_L^G(\chi_3) = \chi_{3^2} - \chi_{42} + \chi_{51} - \chi_6$

Row indexed by $2^21^2$ is $R_L^G(\chi_{21})$ and

Row indexed by $1^6$ is $R_L^G(\chi_{1^3})$. 
Michel Broué’s philosophy

BRAUER=LUSZTIG
References: