Discrete metric spaces: structure and enumeration

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Forb$_n$(H): a history

Notation: Given $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$. $K_n$ = the complete graph on $n$ vertices.

**Definition**

Fix a finite graph $H$ and integer $n$. Define Forb$_n$(H) to be the set of graphs $G$ with the following properties:
- $V(G) = [n]$ and
- $G$ omits $H$ as a (non-induced) subgraph.

**Definition**

Given $l \geq 2$, Col$_n$(l) is the set of $l$-colorable graphs with vertex set $[n]$.

Recall for all $l \geq 2$ and $n$, Col$_n$(l) $\subseteq$ Forb$_n$(K$_{l+1}$).
Case $H = K_3$:

**Theorem (Erdős, Kleitman, Rothschild, 1976)**

1. **Structure:**
   \[
   \lim_{n \to \infty} \frac{|Col_n(2)|}{|Forb_n(K_3)|} = 1.
   \]

2. **Enumeration:**
   \[
   |Forb_n(K_3)| = \left(1 + o\left(\frac{1}{n}\right)\right)|Col_2(n)| = 2^{\frac{1}{2}} \frac{n^2}{2} + o(n^2).
   \]
Forb\textsubscript{n}(H): a history

There are many extensions and generalizations of this to other families of the form Forb\textsubscript{n}(H):

Theorem (Kolaitis, Pr"omel, Rothschild, 1987)

Fix $H = K_{l+1}$ for $l \geq 2$.

1. **Structure:**
   \[ \lim_{n \to \infty} \frac{|Col_n(l)|}{|Forb_n(K_{l+1})|} = 1. \]

2. **Enumeration:** for any $p \geq 1$,
   \[ |Forb_n(K_{l+1})| = \left(1 + o\left(\frac{1}{np}\right)\right)|Col_n(l)| = 2^{(1-\frac{1}{l})}\frac{n^2}{2} + o(1). \]
**Theorem (Prömel, Steger, 1992)**

Suppose \( l \geq 2 \). Suppose \( H \) has \( \chi(H) = l + 1 \) and contains a color-critical edge.

1. **Structure:**

\[
\lim_{n \to \infty} \frac{|Col_n(l)|}{|Forb_n(H)|} = 1.
\]

2. **Enumeration:** for any \( p \geq 1 \),

\[
|Forb_n(H)| = \left( 1 + o\left( \frac{1}{n^p} \right) \right)|Col_n(l)| = 2^{\left( 1 - \frac{1}{l} \right) \frac{n^2}{2} + o(1)}.
\]
Fix an integer \( r \geq 2 \). \( M_r(n) \) is the set of metric spaces with underlying set \([n]\) and distances all in \([r]\).

Given a set \( X \), \( \binom{X}{2} = \{ Y \subseteq X : |Y| = 2 \} \).

Fix an integer \( r \geq 2 \). A simple complete \( r \)-graph is a pair \((V, c)\) where \( V \) is a set of vertices and \( c : \binom{V}{2} \rightarrow [r] \) is a function. \( c \) is called a coloring.

Elements \( G \in M_r(n) \) are naturally simple complete \( r \)-graphs: just color edges \( xy \) with the distance \( d(x, y) \).
A violating triangle is an $r$-graph $H = (V, c)$ with $V = \{x, y, z\}$ such that for some $i, j, k \in [r]$, with $i > j + k$,

$$c(x, y) = i, \ c(x, z) = j, \text{ and } c(y, z) = k.$$

Given two $r$-graphs $G$ and $H$, $G$ omits $H$, if for all injections $f : V(H) \rightarrow V(G)$, there is $xy \in \binom{V(H)}{2}$ such that $c^H(x, y) \neq c^G(f(x), f(y))$.

Observation:
$M_r(n)$ is the set of all simple and complete $r$-graphs $G$ such that

- $V(G) = [n]$,
- $G$ omits all violating triangles.
Questions

1. Given a fixed $r$, what is the asymptotic structure of elements of $M_r(n)$?
2. $|M_r(n)| = ???$
Metric sets

**Definition**

$A \subseteq [r]$ is a *metric set* if for all $a, b, c \in A$, $a \leq b + c$.

**Notation:** Given $s < r \in \mathbb{N}$, $[s, r] = \{s, s + 1, \ldots, r\}$.

**Lemma (Mubayi, T.)**

- When $r \geq 2$ is even, $[\frac{r}{2}, r]$ is a unique largest metric subset of $[r]$.
- When $r \geq 3$ is odd, $[\frac{r-1}{2}, r - 1], [\frac{r+1}{2}, r]$ are the two largest metric subsets of $[r]$.

**Example**

- If $r = 4$, $\{2, 3, 4\}$ is the unique largest metric subset of $[r]$.
- If $r = 5$, $\{2, 3, 4\}$ and $\{3, 4, 5\}$ are the two largest metric subsets of $[r]$.

When $r$ is odd, let $U_r = [\frac{r+1}{2}, r]$ and $L_r = [\frac{r-1}{2}, r - 1]$. 
We now define a special subfamily \( C_r(n) \subseteq M_r(n) \).
Idea: \( C_r(n) \) contains only distances in the “top half” of \([r]\).

**Definition**

When \( r \) is even, \( C_r(n) = \{ G \in M_r(n) : \text{for all} \ a, b \in G, \ d(a, b) \in [\frac{r}{2}, r] \} \).

**Example**

When \( r \) is 4, this is the set of metric spaces on \([n]\) with all distances in \{2, 3, 4\}.

**Definition**

When \( r \) is odd, \( C_r(n) \) is the set of all \( G \in M_r(n) \) with the following property. There is a partition \( P_1, \ldots, P_l \) of \([n]\) such that

- For all \( i \) and \( ab \in \left( \begin{array}{c} P_i \\ 2 \end{array} \right) \), \( d(a, b) \in L_r \).
- For all \( i \neq j \), \((a, b) \in P_i \times P_j, \ d(a, b) \in U_r \).
For example, when $r = 5$, an element of $C_r(n)$ could look like:
Counting $C_r(n)$

Observation:
When $r$ is even $|[\lfloor \frac{r}{2} \rfloor, r]| = \lceil \frac{r+1}{2} \rceil$.
When $r$ is odd, $|L_r| = |U_r| = \lceil \frac{r+1}{2} \rceil$.

When $r \geq 2$ is even,

$$|C_r(n)| = \lceil \frac{r}{2}, r \rceil \binom{n}{2} = \lceil \frac{r+1}{2} \rceil \binom{n}{2}.$$ 

When $r \geq 3$ is odd,

$$\left\lfloor \frac{r+1}{2} \right\rfloor \binom{n}{2} \leq |C_r(n)| \leq n^n \left\lfloor \frac{r+1}{2} \right\rfloor \binom{n}{2} = \left\lfloor \frac{r+1}{2} \right\rfloor \binom{n}{2} + o(n^2).$$
Questions

1. Given a fixed $r$, what is the asymptotic structure of elements of $M_r(n)$?
2. $|M_r(n)| = ???$
Approximate structure

Definition

Given $\delta > 0$ and two elements $G, G' \in M_r(n)$, we say $G$ and $G'$ are $\delta$-close if

$$\left| \left\{ ab \in \begin{pmatrix} [n] \end{pmatrix}_2 : d^G(a, b) \neq d^{G'}(a, b) \right\} \right| \leq \delta n^2.$$ 

Theorem (Mubayi, T.)

For all $r \geq 2$ and $\delta > 0$,

$$\lim_{n \to \infty} \left| \left\{ G \in M_r(n) : G \text{ is } \delta\text{-close to an element of } C_r(n) \right\} \right| \frac{1}{|M_r(n)|} = 1.$$ 

Proof uses multi-color version of Szemeredi’s regularity lemma and a stability theorem.
Corollary (Mubayi, T.)

For all $r \geq 2$, 

$$|M_r(n)| = \left\lceil \frac{r+1}{2} \right\rceil \binom{n}{2} + o(n^2).$$
The even case

Theorem (Mubayi, T.)

When $r$ is even,

$$
\lim_{n \to \infty} \frac{|C_r(n)|}{|M_r(n)|} = 1.
$$

Corollary (Mubayi, T.)

When $r \geq 2$ is even

$$
|M_r(n)| = (1 + o(1))|C_r(n)| = \left\lceil \frac{r + 1}{2} \right\rceil \binom{n}{2} + o(1).
$$
The odd case

**Theorem (Mubayi, T.)**

*When* $r \geq 3$ *is odd,*

$$\lim_{n \to \infty} \frac{|C_r(n)|}{|M_r(n)|} < 1.$$ 

Moreover,

$$|M_r(n)| \geq \left\lceil \frac{r + 1}{2} \right\rceil \binom{n}{2}^{\Omega(n \log n)}.$$
Open questions

When $r$ is odd:
- What is the fine structure of $M_r(n)$?
- $|M_r(n)| = \left\lfloor \frac{r+1}{2} \right\rfloor \binom{n}{2} + \text{??}$.
- What is different about the even and odd cases?
Thank you for listening!