MCS 549 – Mathematical Foundations of Data Science
Fall 2023
Problem Set 1

Lev Reyzin

Due: 10/6/23 at the beginning of class

Instructions: Atop your problem set, please write your name and list your collaborators.

Problems

Prove all your answers.

1. For what value of d is the volume of the d-dimensional unit ball maximized?

2.* Suppose we are given n unit vectors in R^n divided into two sets P, Q with the guarantee that there exists a hyperplane $a \cdot x = 0$ such that every point in P is on one side of it and every point in Q is on the other. Furthermore, assume that the ℓ_2 distance of each point to the hyperplane is at least γ (this is sometimes called the "margin"). Show that a random projection (as defined in the book) to some $c \log n / \gamma^2$ dimensions will have the property that with high probability, the two sets of points will still remain separated by a hyperplane with margin $\gamma/2$.

3. Show that if A is a symmetric matrix with distinct singular values, then the left and right singular vectors are the same and $A = V D V^T$.

4. A Markov chain is said to be symmetric if for all i and j, $p_{ij} = p_{ji}$. What is the stationary distribution of a connected symmetric Markov chain? Prove your answer.

5. What is the hitting time h_{uv} for two adjacent vertices on a cycle of length n? What is the hitting time if edge (u, v) is removed?

6. What is the escape probability of a random walk starting at the root of an infinite binary tree? Show how you arrived at your answer.

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*This problem is extra challenging.