Math 215, Fall 2026, Problem Set 5

0) **[50 pts]** Discuss some topics relevant to this homework (e.g. counting, the inclusion–exclusion principle, the pigeonhole principle, or binomial coefficients) with an LLM of your choice. Ask some interesting questions, seek clarification, and/or probe for more depth. Make the conversation interactive and show that you have given some thought to the LLM's responses. Turn in that chat log as your answer to this question. If you use an LLM for any of the remaining questions on this problem set, also turn in the corresponding chat logs.

1) **[10 pts]** How many integers from 1 through 120 are divisible by at least one of 2, 3, and 5? Show how you use the inclusion–exclusion principle to obtain your answer.

2) **[10 pts]** Let $A_1, A_2, \ldots, A_n$ be finite sets. For each nonempty set $I \subseteq \{1, \ldots, n\}$, write

$$A_I = \bigcap_{i \in I} A_i.$$

Prove by induction on n the full inclusion–exclusion principle:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\varnothing \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} |A_I|.$$

You may use the two-set inclusion–exclusion formula

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

3) **[10 pts]** Let n be a positive integer and suppose that

$$A \subseteq \{1, 2, \ldots, 2n\} \qquad \text{and} \qquad |A| = n + 1.$$

Use the pigeonhole principle to prove that A contains two consecutive integers.

4) **[10 pts]** Let X and Y be finite sets with

$$|X| = 4, \qquad |Y| = 6.$$

(a) How many functions $f : X \to Y$ are there?

(b) How many of these functions are injections?

(c) How many are not injections?

Explain your answers.

5) **[10 pts]** Let X be a set with n elements. Count in two different ways the number of pairs (A, x) such that

$$A \subseteq X \qquad \text{and} \qquad x \in A,$$

and use your two counts to prove that

$$\sum_{k=0}^{n} k \binom{n}{k} = n 2^{n-1}.$$